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Continua with microstructure modelled by the geometry of higher-order contact

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Abstract

In the paper, the Finslerian-geometry-oriented model of the continuum with microstructure is formulated within the frame of Newtonian–Eshelbian continuum mechanics, based on the information characterizing a structure-dependent evolution of state variables. In this approach, position- and direction-dependent deformation and strain measures are used to describe the motion of the continuum with microstructure at the macro- and microlevel. The variational arguments for a Lagrangian functional defined on the Finslerian bundle are used to derive dynamic balance laws, boundary and transversality conditions for macro- and microstresses of deformational and configurational type. The dissipation inequality for the thermo-inelastic deformation processes is formulated by the sufficiency condition of Weierstrass type for the action integral. The presented geometric technique is illustrated in the following examples. The damage tensor, identified with a measure of reduction of load carrying area elements caused by the development of microcracks or microvoids, is defined on the tangent bundle using the lifting technique. The macro–micro constitutive equations and the associated phenomenological constitutive relations for the thermo-inelastic processes are derived in terms of the free energy functional and a dissipation potential. A strain-induced crack propagation criterion, defined by the difference between the strain energy release rate and the rate of the surface energy of the crack, is formulated for the kinking of cracks.

Keywords: Microstructure model; Geometry; Higher-order contact

1. Introduction

I am very pleased to present a paper for this special issue of the International Journal of Solids and Structures dedicated to Multifield Theories. As my topic, I have chosen to discuss a general geometrical approach within the frame of Newtonian–Eshelbian continuum mechanics, which leads directly to the complete set of equations for a dissipative model of continuum with microstructure. One of the reasons for this choice is the fact that a detailed description of inelastic material behaviour very often demands

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combined theoretical and experimental approaches. At the continuum level, the description of any phenomenon involving inelastic, hysteretic and rate dependent material response combined with nucleation and evolution of defects requires an extended theoretical methodology rather than a classical mechanical description. In order to deal with a heterogeneous nature of stressed (distorted) state of the body, where the motion of material particles (extended objects) is influenced by the presence of dislocations, cracks, voids and other defects, a multidimensional configuration space is required for its description. The present study provides the theoretical, structure-dependent framework for the continuum mechanics methodology in a form suitable to model deformation processes of continua with microstructure based on a unified Finslerian geometry conception. Mostly based on recent works in this subject, the contents of this paper show how the Finslerian geometry (Finsler, 1918; Cartan, 1934), the classical case of the higher-order contact geometry (Kawaguchi, 1962; Yano and Ishihara, 1973; Miron, 1997), can be used to find a complete set of field equations for the continuum with microstructure at the macro- and microlevel. This generalized metric geometry is a natural extension of the Riemannian geometry, where a metric tensor depends both on the position and the direction (Rund, 1959; Asanov, 1985; Matsumoto, 1986; Abate and Patrizio, 1994; Antonelli and Miron, 1996; Bejancu, 1990; Miron, 1997). Certain geometric aspects of the geometry of higher-order contact can be formulated using the lifting technique (Yano and Ishihara, 1973). Because a link between the geometry of higher-order contact and the lifting technique manifests itself in different ways, its essence is only illustrated here (cf. Saczuk, 1992, 1994; Fu et al., 1998). A presentation which follows is strictly correlated with the role played by defects in the mechanical behaviour of materials since, e.g., deformation and fracture are not exceptions in determining whether or not particular processes are likely to occur.

A fundamental problem in continua with microstructure accounting for defect distributions in the bulk, originated independently by Kondo (1952, 1955) and Bilby et al. (1955), was developed by Bilby (1960), Eshelby (1951, 1961), Kröner (1955, 1958) and Seeger (1961) in terms of the continuous distribution of dislocations. Much efforts has been expended on the development of concepts of continua with microstructure based on notions of continuum mechanics (Truesdell and Toupin, 1960; Truesdell and Noll, 1965; Noll, 1967; Wang, 1967; Eringen and Kadafar, 1976; Green and Rivlin, 1964; Mandel, 1974; Mindlin, 1964; Capriz, 1989; Capriz and Podio-Guidugli, 1976; Green and Naghdi, 1995; Le and Stumpf, 1996a,b; Naghdi and Srinivasa, 1993a,b). The origin of such theories is usually traced back to Cosserat and Cosserat (1909), who introduced a rotation tensor at each material point as an additional field variable. The direction adopted by Naghdi and Srinivasa (1993a,b) and Le and Stumpf (1996a,b) and based on a director concept, was generalized by Stumpf and Saczuk (2000). Recently, the Finslerian geometry has been applied by Saczuk (1996, 1997a,b) to propose a continual description of inelastic deformations of solids with microstructure.

A degradation process of the mechanical properties of solids, documented by experimental results, is generally connected with nucleation and evolution, growth and coalescence of micro- and macrodefects as voids and cracks. Such defects evolve, to some degree, independently to the motion of the body due to the configurational forces (first derived and discussed by Eshelby, 1951), whose divergence embody the inhomegeneity of the body. The role played by configurational forces on (macro)defects has attracted increased interest in the recent years (Stumpf and Le, 1990, 1992; Maugin and Trimarco, 1992; Maugin, 1993; Gurtin, 1995; Gurtin and Podio-Guidugli, 1996). A physically justified description of a progressive degradation of the bulk properties of solids with defects was first proposed by Kachanov (1958) within a scalar model of isotropic damage. Several investigations, following Kachanov (1958), have proposed modifications of the scalar model (Kachanov, 1986; Chaboche, 1988a; Krajcinovic, 1989; Lemaitre, 1992). Damage accounting for finite elasto-plastic deformations is further investigated among others by Simo and Ju (1987) and Lubarda and Krajcinovic (1995). A fibre bundle approach to an anisotropic damage analysis is presented by Fu et al. (1998), where the damage state is identified with a breakdown of the holonomicity in the continuum.

The evolution of cracks in solids, crucial to the fracture behaviour, is generally connected with the non-stable process of the propagation of displacement discontinuities and with the formation of new boundaries that develop in the medium. Cracks dissipate energy by creating new surfaces, and the fracture toughness of the material is related to the energy release rate at the tip of cracks. Numerous investigations (e.g., Gilman, 1960; Averbach, 1965; Kitagawa et al., 1975; Gurson, 1977) have shown the role of microstructure, environment and crack size influencing the fracture response of materials. The response of the crack on loads is realized by structural changes in its orientation and topography in order to attain the infimum of the free energy state (Francfort and Marigo, 1998). Most of the analysis of fracture mechanics is devoted to the determination of the energy release rate, initiated by Griffith (1920), for various special conditions (cf. Hertzberg, 1983; Freund, 1990). Standard models in fracture mechanics are based on the assumption that the total energy of the body is the sum of a bulk term, representing the strain energy, and of a surface term, representing the energy associated with the displacement discontinuity. Using the energy balance and the singularity of the stress field in the vicinity of the crack tip, they lead to the finite energy release rate as the crack expands. Of particular significance is the mathematical concept for a J-integral fracture analysis established by Rice (1968).

The present paper is concerned with the development of the Finslerian-geometry-oriented model of the continuum with microstructure. A kinematical concept used in this paper is largely motivated by features of orientation-dependent phenomena like shear bands and cracks. The assumption relating the position of material points in the actual configuration in terms of a position-direction dependent function leads to deformation and strain measures for macro- and micromotion all depending on the position and direction (Section 2). The variational arguments for a Lagrangian functional defined on the Finslerian bundle and an assumed one-parameter family of transformations of the independent and dependent variables are used to derive dynamic balance laws, boundary and transversality conditions for macro- and microstresses of deformational as well as configurational type, where the latter have to be satisfied by the driving forces on macro- and microdefects (Sections 3.1 and 3.2). The dissipation inequality for the thermo-inelastic deformation processes is formulated by the sufficiency condition of Weierstrass type for the action integral (Section 3.3). The presented geometric approach is illustrated in the following examples. The damage tensor, identified with a measure of reduction of load carrying area elements caused by the development of microcracks or microvoids, is defined on the tangent bundle using the so-called lifting technique. The resulting damage tensor is composed either from initial, additional and direct (deformation-induced) damages or from direct and transferred ones (Section 4.1). The macro–micro constitutive equations and the associated phenomenological constitutive relations for the thermal-inelastic processes and a constitutive damage model of Kachanov's type accounting for the crack density are derived in terms of the free energy functional and a dissipation potential (Section 4.2). A strain-induced crack propagation criterion, defined by the difference between the strain energy release rate and the rate of the surface energy of the crack, is formulated for the kinking of cracks (Section 4.3).

2. Basic kinematic equations

We consider a material body (continuum) $\mathscr{B}$ with microstructure at the equilibrium configuration C_0 in which the density ρ_0 and the temperature θ_0 have the uniform values, the stress state is not uniform and the heat flux is everywhere zero. We will refer to C_0 as the global reference configuration and denote by C_t the configuration attained by $\mathscr{B}$ at the current time t. The body $\mathscr{B}$ is considered to be modelled by a generalized oriented continuum endowed at each point with a deformable vector. Such a choice stems form the fact that the motion of a material particle (an extended geometric object) in $\mathscr{B}$ from one equilibrium configuration to another one involves the cooperative motion of many microingredients (dislocations, microcracks) and requires for its detailed description a multidimensional configuration space (cf. Kunin, 1990). The oriented particles of $\mathscr{B}$ are identified by their position $\mathbf{x}$ and direction $\mathbf{y}$, briefly denoted by the pair (position, direction) $=(\mathbf{x},\mathbf{y})$ in the reference configuration C_0.

Within the adopted methodology, each material point of $\mathscr{B}$ located at a position $\mathbf{x}$ is endowed with an internal structure described by an additional independent microvector (director) field $\mathbf{y}$. All such points are embedded in the continuum which contains heterogeneously distributed dislocations and voids (crack dislocation arrays). [2] The state space B for the body without microstructure is the three-dimensional physical Euclidean space. The state space $\mathscr{B}$ for the body with microstructure is the fibre space

$$B \times \times M \subset \mathbb{E}^3 \times \mathbb{E}^3,$$

where B is the body and M, the microstructure in the reference configuration C_0, $\mathbb{E}$, the Euclidean space and $\times\times$ denotes the local Cartesian product (a local trivialization).

We assume that a deformation of the body $\mathscr{B}$ can be expressed in terms of the position-director-dependent deformation vector $\boldsymbol{\chi}$ relating particles in the actual configuration C_t by means of a smooth invertible map

$$\mathbf{X} = \boldsymbol{\chi}(\mathbf{x},\mathbf{y}), \qquad \boldsymbol{\chi} : \mathscr{B} \to \mathbb{E}^3 \times \mathbb{E}^3 \tag{2.1a, b}$$

in terms of oriented particles in the reference configuration C_0. In the manifold-theoretic setting, we shall therefore consider a vector field $\boldsymbol{\chi}$ on $\mathscr{B}$, which can give rise to a vector field $\bar{\boldsymbol{\chi}}$ on the base space B. The notion of deformation is here identified with an injection $\boldsymbol{\chi}$ of $\mathscr{B}$ into $\mathbb{E}^3 \times \mathbb{E}^3$.

As a special case of Eq. (2.1), the kinematics of a generalized Cosserat continuum can be described by two smooth functions

$$\mathbf{X} = \phi(\mathbf{x}), \tag{2.2}$$

$$\mathbf{Y} = \mathscr{D}(\mathbf{x})\mathbf{y}, \tag{2.3}$$

where $\mathbf{y} = (y^1, y^2, y^3)$ is the director in the initial configuration and $\mathbf{Y} = (Y^1, Y^2, Y^3)$ the director in the actual configuration, and $\mathscr{D}$ denotes a linear map. From the geometric point of view, the two functions (2.2) and (2.3) represent, in reality, coordinate transformations (extended point transformations) in $\mathbb{E}^3 \times \mathbb{E}^3$.

Formally, the motion of the body $\mathscr{B}$ is a family of time-dependent diffeomorphisms $\boldsymbol{\chi}_t$, i.e. the time-dependent relation of Eq. (2.1a),

$$\mathbf{X}(t) = \boldsymbol{\chi}_t(\mathbf{x},\mathbf{y}) = \boldsymbol{\chi}(\mathbf{x},\mathbf{y},t), \tag{2.4}$$

where

$$\boldsymbol{\chi} : \mathscr{B} \times \mathbb{R} \to \mathbb{E}^3 \times \mathbb{E}^3 \times \mathbb{R} \tag{2.5}$$

is the time-dependent analogue of Eq. (2.1a).

2.1. Internal state specification

In this section, we discuss basic concepts suitable for defining the internal state of the body $\mathscr{B}$. This step, which has no close correlation with the continuum mechanics methodology, is adapted from the theory of non-holonomic subspaces (Yano and Davies, 1955; Deicke, 1955; Schouten, 1954) to introduce the microstructure-dependent covariant operators on $\mathscr{B}$. We assume that the internal state of the body can be defined by specifying, for instance, the dislocation functional $W(\mathbf{x},\mathbf{y}) \equiv L^2(\mathbf{x},\mathbf{y})$, where $L = L(\mathbf{x},\mathbf{y})$ is the fundamental function describing the generation and evolution of dislocations. Accordingly, since cracks are

[2] In this heterogeneous medium, classical particles can be viewed as individual continua.

clearly responsible for the fracture of materials in a wide range of situations, the restriction of W to the dislocation functional is not a severe limitation. This functional represents, very often, an anharmonic approximation of interactions between dislocations and their self-energy. The mentioned identification is motivated by the assumption in materials science saying that the energy density of dislocations is proportional to the square of their Burgers vector (cf. Nabarro, 1967; Hirth and Lothe, 1968). The internal vector field $\mathbf{y}$ can be identified here with the Burgers vector $\mathbf{b}$ multiplied by a number of mobile dislocations (Eshelby et al., 1951).

For this purpose, we need to consider an arbitrary six-dimensional Riemannian space V with the field of frames $(\mathbf{g}_k, \mathbf{z}_k)$. The natural frame $(\mathbf{g}_k, \mathbf{z}_k)$ of V is not transformed as basis, if the coordinate transformation (2.2) and (2.3) are adopted to this space. This fact suggests to regard the space under consideration as a non-holonomic subspace (here $\mathscr{B}$) of the space V. This non-holonomic subspace has the frame of reference $({}^{\mathrm{h}}\mathbf{g}_k, \mathbf{z}_k)$, defined below, which is not associated with the coordinate system on $\mathscr{B}$. The solution of this problem will be based on the introduction of non-holonomic frames in V, instead of $\mathbf{g}_k$ and $\mathbf{z}_k$, which leads in a natural manner to connection coefficients in the space V (cf. Yano and Davies, 1955).

According to Eqs. (2.2) and (2.3), the local bases $(\mathbf{g}_k, \mathbf{z}_k)$ at a point $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{g}_{k'}, \mathbf{z}_{k'})$ at a point $(\mathbf{x}', \mathbf{y}')$ are related by

$$\begin{bmatrix} \mathbf{g}_k \\ \mathbf{z}_k \end{bmatrix} = \begin{bmatrix} \partial\phi^{k'}/\partial x^k & (\partial D_l^{k'}/\partial x^k) y^l \\ 0 & D_k^{k'} \end{bmatrix} \begin{bmatrix} \mathbf{g}_{k'} \\ \mathbf{z}_{k'} \end{bmatrix}, \tag{2.6}$$

where $y^{k'} = D_k^{k'}(\mathbf{x}) y^k$, $\phi^{k'} = \phi^k \circ \boldsymbol{\chi}^{-1}$, $x^{k'} = x^k \circ \boldsymbol{\chi}^{-1}$, etc. As follows from Eq. (2.6), the vectors $\mathbf{g}_k$ are not transformed as vectors. Suppose now that there exists on $\mathscr{B}$ a family of functions $N_j^k(\mathbf{x}, \mathbf{y})$, called coefficients of a non-linear connection, with the following law of transformation (cf. Čomić, 1989):

$$N_j^k(\mathbf{x}, \mathbf{y}) = N_{l'}^{k'}(\mathbf{x}', \mathbf{y}') \frac{\partial \phi^{l'}}{\partial x^j} D_{k'}^k(\mathbf{x}) - \frac{\partial D(\mathbf{x}')_{k'}^k}{\partial x^{l'}} \frac{\partial \phi^{l'}}{\partial x^j} y^{k'}. \tag{2.7}$$

The existence of the non-linear connection $\mathbf{N}_\Gamma(N_l^k)$ allows one to introduce an adapted basis $({}^{\mathrm{h}}\mathbf{g}_k, \mathbf{z}_k)$ in V, where the vectors ${}^{\mathrm{h}}\mathbf{g}_k$ defined by

$${}^{\mathrm{h}}\mathbf{g}_k = \mathbf{g}_k - N_k^l \mathbf{z}_l \tag{2.8}$$

are transformed under Eqs. (2.2) and (2.3) as vectors. The field of bases $({}^{\mathrm{h}}\mathbf{g}_k, \mathbf{z}_k)$, in turn, is adapted to the decomposition of the tangent space $T\mathscr{B}$ into the subbundle $(T\mathscr{B})^{\mathrm{h}}$ spanned by ${}^{\mathrm{h}}\mathbf{g}_k$ and the subbundle $(T\mathscr{B})^{\mathrm{v}}$ spanned by $\mathbf{z}_k$ to model the macrobehaviour and the microbehaviour of the oriented material particles, respectively.

The frame $(\mathbf{g}^k, {}^{\mathrm{v}}\mathbf{z}^k)$ dual to $({}^{\mathrm{h}}\mathbf{g}_k, \mathbf{z}_k)$ is then defined by

$$\mathbf{g}^k = dx^k, \tag{2.9a}$$

$${}^{\mathrm{v}}\mathbf{z}^k = \mathbf{z}^k + N_j^k \mathbf{g}^j, \tag{2.9b}$$

where $\mathbf{z}^k = dy^k$. In dual terms, the cotangent space $T^\star\mathscr{B}$ is decomposed into the subbundle $(T^\star\mathscr{B})^{\mathrm{h}}$ spanned by $\mathbf{g}^k$ and the subbundle $(T^\star\mathscr{B})^{\mathrm{v}}$ spanned by ${}^{\mathrm{v}}\mathbf{z}^k$.

The specified quantities N_l^k are defined by the fundamental function, [3] L, which from a physical point of view can be identified with the free energy of dislocations and substructures induced by the deformation

[3] Note that the non-linear connection coefficients can be determined by coefficients of a spray on manifold (cf. Sternberg, 1964; Miron, 1997).

process. In this way, one can define in an invariant manner a local topology (a topography of element glide resistance, Kocks et al., 1975) of the dislocated body within the frame of macro- and microspace. To define macro- and microconnections in the body $\mathscr{B}$, which are intended to describe the internal state of it, we proceed as follows.

We introduce covariant operators ∇^{h} and ∇^{v} for the macro- and microparts of the tangent space $T\mathscr{B}$ using Christoffel symbol concepts as follows:

$$\nabla^{\mathrm{h}}_{^h\mathbf{g}_i} {}^{\mathrm{h}}\mathbf{g}_j = {}^{\mathrm{h}}\Gamma^k_{ij}\,{}^{\mathrm{h}}\mathbf{g}_k, \tag{2.10}$$

$$\nabla^{\mathrm{v}}_{\mathbf{z}_i} \mathbf{z}_j = {}^{\mathrm{v}}\Gamma^k_{ij}\mathbf{z}_k. \tag{2.11}$$

Connection coefficients of $\mathbf{\Gamma}^{\mathrm{h}}$ $({}^{\mathrm{h}}\Gamma^k_{ij})$ and $\mathbf{\Gamma}^{\mathrm{v}}$ $({}^{\mathrm{v}}\Gamma^k_{ij})$, compatible with the metric tensor g_{ij}, (2.16a), specified by the conditions

$$\nabla^{\mathrm{h}}_{^h\mathbf{g}_k} g_{ij} = \delta_k g_{ij} - g_{lj}{}^{\mathrm{h}}\Gamma^l_{ik} - g_{il}{}^{\mathrm{h}}\Gamma^j_{jk} = 0, \tag{2.12}$$

$$\nabla^{\mathrm{v}}_{\mathbf{z}_k} g_{ij} = \bar{\partial}_k g_{ij} - g_{lj}{}^{\mathrm{v}}\Gamma^l_{ik} - g_{il}{}^{\mathrm{v}}\Gamma^l_{jk} = 0 \tag{2.13}$$

are represented by the generalized Christoffel symbols [4]

$${}^{\mathrm{h}}\Gamma^i_{jk} = \tfrac{1}{2}g^{il}\left(\delta_j g_{lk} + \delta_k g_{jl} - \delta_l g_{jk}\right), \tag{2.14}$$

$${}^{\mathrm{v}}\Gamma^i_{jk} = \tfrac{1}{2}g^{il}(\bar{\partial}_j g_{lk} + \bar{\partial}_k g_{jl} - \bar{\partial}_l g_{jk}). \tag{2.15}$$

The differentiation operator δ_k used in Eq. (2.14) stands for $\delta_k = \partial_k - N^l_k\bar{\partial}_l$, where $\partial_k = \partial/\partial x^k$ and $\bar{\partial}_k = \partial/\partial y^k$.

The introduced connection coefficients ${}^{\mathrm{h}}\Gamma^i_{jk}, {}^{\mathrm{v}}\Gamma^i_{jk}$ and N^l_k are calculated from the assumed dislocation (microstructure-dependent) functional $L^2 = W(\mathbf{x}, \mathbf{y})$ in terms of the metric tensor

$$g_{ij}(\mathbf{x}, \mathbf{y}) = \tfrac{1}{2}\bar{\partial}_i\bar{\partial}_j L^2(\mathbf{x}, \mathbf{y}), \tag{2.16a}$$

$$L^2(\mathbf{x}, \mathbf{y}) = g_{ij}(\mathbf{x}, \mathbf{y})y^i y^j \tag{2.16b}$$

and using the geodesic equations (the first-order evolution equations for the internal state vector $\mathbf{y}$) (cf. Miron, 1997)

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} + 2G^i(\mathbf{x}, \mathbf{y}) = 0, \tag{2.17}$$

where the components of the contravariant vector G^i are defined by

$$2G^i(\mathbf{x}, \mathbf{y}) = \frac{1}{2}g^{ij}\left(\frac{\partial^2 L^2}{\partial y^j \partial x^k}y^k - \frac{\partial L^2}{\partial x^j}\right). \tag{2.18}$$

The restrictions on the fundamental function L (positively homogeneous of degree one with respect to $\mathbf{y}$) are essentially those needed to ensure the regularity of the minimization problem of the integral $\int L(\mathbf{x}, \mathrm{d}\mathbf{x})$. An example of a position–direction-dependent functional $L^2 = W(\mathbf{x}, \mathbf{y})$ describing the energy of dislocations can be found in Stumpf and Saczuk (2000).

According to conditions (2.12) and (2.13), and using the methodology of the Finslerian geometry (Rund, 1959; Matsumoto, 1986), connection coefficients (2.14) and (2.15) are finally reduced to

[4] Originally denoted by $\Gamma^{\star i}_{jk}$ and C^i_{jk}, respectively; cf. Cartan (1934) and Rund (1959).

$$ {}^{\mathrm{h}}\Gamma_{ijk} = g_{jl}{}^{\mathrm{h}}\Gamma^{l}_{ik}, \qquad {}^{\mathrm{h}}\Gamma_{ijk}(\mathbf{x},\mathbf{y}) = \gamma_{ijk} - {}^{\mathrm{v}}\Gamma_{kjl}N^{l}_{i} - {}^{\mathrm{v}}\Gamma_{ijl}N^{l}_{k} + {}^{\mathrm{v}}\Gamma_{ikl}N^{l}_{j}, \tag{2.19} $$

where

$$ \gamma_{ijk}(\mathbf{x},\mathbf{y}) = \tfrac{1}{2}(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}), \qquad \gamma_{ijk} = g_{jl}\gamma^{l}_{ik}, \tag{2.20} $$

$$ {}^{\mathrm{v}}\Gamma_{ijk} = g_{jl}{}^{\mathrm{v}}\Gamma^{l}_{ik}, \qquad {}^{\mathrm{v}}\Gamma_{ijk}(\mathbf{x},\mathbf{y}) = \tfrac{1}{2}\dot{\partial}_k g_{ij}(\mathbf{x},\mathbf{y}). \tag{2.21} $$

The tensor ${}^{\mathrm{v}}\Gamma_{ijk}$, called the Cartan tensor field, is positively homogeneous of degree -1 with respect to $\mathbf{y}$, totally symmetric and satisfies the following property:

$$ {}^{\mathrm{v}}\Gamma_{ijk}y^{i} = {}^{\mathrm{v}}\Gamma_{ijk}y^{j} = {}^{\mathrm{v}}\Gamma_{ijk}y^{k} = 0 \tag{2.22} $$

together with

$$ \partial_l{}^{\mathrm{v}}\Gamma_{ijk}y^{i} = \partial_l{}^{\mathrm{v}}\Gamma_{ijk}y^{j} = \partial_l{}^{\mathrm{v}}\Gamma_{ijk}y^{k} = 0. $$

The spray of the non-linear connection (2.18) written in terms of the connections γ^{i}_{jk} and ${}^{\mathrm{h}}\Gamma^{i}_{jk}$ has the form

$$ G^{l}(\mathbf{x},\mathbf{y}) = \tfrac{1}{2}\gamma^{l}_{jk}y^{j}y^{k}, \qquad N^{l}_{k}(\mathbf{x},\mathbf{y}) = \bar{\partial}_k G^{l} = {}^{\mathrm{h}}\Gamma^{l}_{jk}y^{j}. \tag{2.23} $$

One should stress here the evolution character of the internal vector field $\mathbf{y}$ which results from the fact that the vector field $\mathbf{y}$ being the infinitesimal generator induces the flow $\phi(\epsilon,\mathbf{x})$ (cf. Olver, 1993) in the internal state space. The flow is, by definition, the parameterized (here by ϵ) integral curve passing through $\mathbf{x}$ in $\mathscr{B}$ defined by the following differential problem:

$$ \mathbf{y} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\phi(\epsilon,\mathbf{x}), \qquad \phi(\epsilon_0,\mathbf{x}) = \mathbf{x} $$

for all ϵ (for further details see Olver, 1993). This differential problem states that $\mathbf{y}$ is tangent to the curve $\phi(\epsilon,\mathbf{x})$ for fixed $\mathbf{x}$, i.e., for the specified initial conditions for this curve.

2.2. Deformation gradients

A deformation gradient in the generalized oriented continuum $\mathscr{B}$ is defined in terms of covariant derivatives (2.10) and (2.11) as follows. First, we introduce the direct sum of the covariant derivatives

$$ \nabla^{\mathrm{h}} + \nabla^{\mathrm{v}} = \triangledown(\nabla^{\mathrm{h}} \oplus \nabla^{\mathrm{v}})\Delta, $$

where $\triangledown$ is the codiagonal operator (cf. Pareigis, 1970) and $\oplus$ the direct sum. Using the matrix notation for the diagonal operator Δ, the codiagonal operator $\triangledown$ and the direct sum of covariant derivatives $\nabla^{\mathrm{h}} \oplus \nabla^{\mathrm{v}}$, we get

$$ \Delta = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}, \quad \triangledown = (\mathbf{1}\,\mathbf{1}), \quad \nabla^{\mathrm{h}} \oplus \nabla^{\mathrm{v}} = \begin{pmatrix} \nabla^{\mathrm{h}} & 0 \\ 0 & \nabla^{\mathrm{v}} \end{pmatrix}, $$

where $\mathbf{1}$ is the identity tensor on $\mathscr{B}$. Thus, the addition $\nabla^{\mathrm{h}} + \nabla^{\mathrm{v}}$ is identified here with the following composition:

$$ \nabla^{\mathrm{h}} + \nabla^{\mathrm{v}} = (\mathbf{1}\,\mathbf{1})\begin{pmatrix} \nabla^{\mathrm{h}} & 0 \\ 0 & \nabla^{\mathrm{v}} \end{pmatrix}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}. \tag{2.24} $$

According to Eq. (2.24), the deformation gradient $\mathbf{F}$ in the generalized oriented continuum $\mathscr{B}$ is expressed by

$$ \mathbf{F} = \left(\nabla^{\mathrm{h}} + \nabla^{\mathrm{v}}\right)\mathbf{X} = \mathbf{F}^{\mathrm{h}} + \mathbf{F}^{\mathrm{v}}, \tag{2.25} $$

where [5]

$$\mathbf{F}^{\mathrm{h}} = \nabla^{\mathrm{h}}\mathbf{X} = (\mathbf{F}^{\mathrm{h}})^{a}_{k}\,{}^{\mathrm{h}}\bar{\mathbf{g}}_{a} \otimes \mathbf{g}^{k}, \qquad \mathbf{F}^{\mathrm{v}} = \nabla^{\mathrm{v}}\mathbf{X} = (\mathbf{F}^{\mathrm{v}})^{a}_{k}\bar{\mathbf{z}}_{a} \otimes {}^{\mathrm{v}}\mathbf{z}^{k}, \tag{2.26}$$

are macro- and micropart of $\mathbf{F}$, $({}^{\mathrm{h}}\bar{\mathbf{g}}_a, \bar{\mathbf{z}}_a)$ is the basis in the actual configuration C_{t} and $\otimes$ denotes the tensor product. The additive decomposition (2.25), opposite to the multiplicative one assumed in the classical plasticity (Eckart, 1948; Lee, 1969), does not demand here any additional assumptions. The components $\mathbf{F}^{\mathrm{h}}$ and $\mathbf{F}^{\mathrm{v}}$ do not describe strictly regular (say, elastic) and irregular (say, plastic) phenomena. It is apparent, therefore, that the inelastic behaviour of a solid (a simple case of the dissipation phenomenon) cannot be treated as a simple superposition of regular and irregular constituents.

Components of the deformation gradients $\mathbf{F}^{\mathrm{h}}$ and $\mathbf{F}^{\mathrm{v}}$ in Eq. (2.26) are given by

$$(\mathbf{F}^{\mathrm{h}})^{a}_{k} = \partial_k X^a - \bar{\partial}_l X^a \bar{\partial}_k G^l + {}^{\mathrm{h}}\Gamma^{a}_{bk} X^b, \tag{2.27}$$

$$(\mathbf{F}^{\mathrm{v}})^{a}_{k} = \bar{\partial}_k X^a + {}^{\mathrm{v}}\Gamma^{a}_{bk} X^b \tag{2.28}$$

with

$${}^{\mathrm{h}}\Gamma^{a}_{bk} = \delta^{a}_{i}\delta^{j}_{b}({}^{\mathrm{h}}\Gamma^{i}_{jk} \circ \boldsymbol{\chi}^{-1}), \qquad {}^{\mathrm{v}}\Gamma^{a}_{bk} = \delta^{a}_{i}\delta^{j}_{b}({}^{\mathrm{v}}\Gamma^{i}_{jk} \circ \boldsymbol{\chi}^{-1}).$$

The connection coefficients appearing in Eqs. (2.27) and (2.28) are defined in terms of components of the metric tensor $\mathbf{g} = \mathbf{g}(\mathbf{x}, \mathbf{y})$ according to Eqs. (2.14), (2.15) and (2.23).

We understand that the elimination of the internal vector field $\mathbf{y}$ from Eqs. (2.17) and (2.26)–(2.28) leads to a time-dependent relation for the deformation gradient $\mathbf{F}$.

There are a number of special cases allowing to simplify the representation of $\mathbf{F}$ depending on whether (i) the internal state variables are neglected and/or (ii) $\mathbf{X}$ is a function of $\mathbf{x}$ or $\mathbf{y}$ or both $\mathbf{x}$ and $\mathbf{y}$. For instance, if $\mathbf{X} = \mathbf{X}(\mathbf{x})$, then

$$(\mathbf{F}^{\mathrm{h}})^{a}_{k} = \partial_k X^a + {}^{\mathrm{h}}\Gamma^{a}_{bk} X^b \quad \text{and} \quad (\mathbf{F}^{\mathrm{v}})^{a}_{k} = {}^{\mathrm{v}}\Gamma^{a}_{bk} X^b.$$

One should note that whenever a given state of $\mathscr{B}$ has associated with it a non-vanishing torsion tensor, then this state contains dislocations, as cogently argued by Kondo (1955). This fact implies that in the case of $\mathscr{B}$, the objects ${}^{\mathrm{v}}\Gamma^{a}_{bk}$ and $N^{i}_{k} = \bar{\partial}_k G^i$ are non-singular. If the condition ${}^{\mathrm{v}}\Gamma^{a}_{bk} = 0$ is satisfied, then the vertical part of deformation is Euclidean and the dissipative character of this measure is lost.

It is natural to assume that the deformation $\boldsymbol{\chi}$ is an orientation-preserving diffeomorphism demanding

$$J = \det \mathbf{F} = J^{\mathrm{h}} J^{\mathrm{v}} > 0 \tag{2.29}$$

with $J^{\mathrm{h}} = \det \mathbf{F}^{\mathrm{h}}$ and $J^{\mathrm{v}} = \det \mathbf{F}^{\mathrm{v}}$. For mappings which have continuous derivatives, this is the necessary and sufficient condition for invertibility. Since $\mathbf{F}$ is invertible, one can use the polar decomposition from linear algebra (Chevalley, 1946), and uniquely decompose $\mathbf{F}$ as follows:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{F}^{\mathrm{h}} + \mathbf{F}^{\mathrm{v}}, \tag{2.30}$$

where

$$\mathbf{F}^{\mathrm{h}} = \mathbf{R}^{\mathrm{h}}\mathbf{U}^{\mathrm{h}}, \qquad \mathbf{F}^{\mathrm{v}} = \mathbf{R}^{\mathrm{v}}\mathbf{U}^{\mathrm{v}}. \tag{2.31}$$

Introduced above macro- and microstretch tensors, $\mathbf{U}^{\mathrm{h}}$ and $\mathbf{U}^{\mathrm{v}}$, are the positive definite tensors and macro- and microrotation tensors, $\mathbf{R}^{\mathrm{h}}$ and $\mathbf{R}^{\mathrm{v}}$, the proper orthogonal tensors. By virtue of condition (2.29), relation

[5] Objects connected with the reference state of $\mathscr{B}$ are denoted by lowercase Latin letters that occupy the centre of the alphabet, that is $i, j, k, \ldots$, while those connected with the actual state are designated by lowercase Latin letters at the front of the alphabet, that is $a, b, c, \ldots$

(2.4) is invertible for any fixed value of t. This fact enables us to express various fields (such as the displacement and the temperature) as the functions of $(\mathbf{x}, \mathbf{y}, t)$.

Similarly, the velocity and acceleration of the material point at time t are defined in the standard manner,

$$\mathbf{v}_t = \mathbf{v}(\mathbf{x}, \mathbf{y}, t) = \dot{\mathbf{X}}(\mathbf{x}, \mathbf{y}, t) \equiv \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{x}, \mathbf{y}, t), \tag{2.32}$$

$$\mathbf{a}_t = \mathbf{a}(\mathbf{x}, \mathbf{y}, t) = \ddot{\mathbf{X}}(\mathbf{x}, \mathbf{y}, t) \equiv \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{x}, \mathbf{y}, t). \tag{2.33}$$

Here, the dot is used to denote the partial time derivative under $(\mathbf{x}, \mathbf{y})$ fixed.

2.3. Material strain measures

Before formulating any definition of strain, it is necessary to define a distance in the space modelling the behaviour of the body $\mathscr{B}$. An element of length of $\mathscr{B}$ can be defined in the reference configuration C_0 as

$$(ds)^2 = g_{kl}(\mathbf{x}, \mathbf{y})dx^k dx^l + g_{kl}(\mathbf{x}, \mathbf{y})Dy^k Dy^l$$

and, in an analogical manner, in the actual configuration C_t,

$$(d\bar{s})^2 = \bar{g}_{ab}(\mathbf{x}, \mathbf{y})d\bar{x}^a d\bar{x}^b + \bar{g}_{ab}(\mathbf{x}, \mathbf{y})D\bar{y}^a D\bar{y}^b,$$

where g_{kl} and $\bar{g}_{ab}$ are components of the metrics, while dx^k, Dy^k and $d\bar{x}^a, D\bar{y}^a$ are the components of length measured in the reference and the actual configuration of $\mathscr{B}$. To formulate an exact definition of inelastic strain in $\mathscr{B}$ we use definition (2.25) and write

$$(d\bar{s})^2 - (ds)^2 = \mathbf{F}^\mathrm{T}\bar{\mathbf{g}}\mathbf{F} - \mathbf{g} = \mathbf{C} - \mathbf{g} = 2\mathbf{E}. \tag{2.34}$$

The measure of deformation $\mathbf{C} : T^\star\mathscr{B} \to T^\star\mathscr{B}$ introduced here is a structure-dependent Cauchy–Green type strain measure defined by

$$\mathbf{C} = \mathbf{F}^\mathrm{T}\bar{\mathbf{g}}\mathbf{F} = \mathbf{C}^\mathrm{h} + \mathbf{C}^\mathrm{v},$$

where $\mathbf{C}^\mathrm{h} = (\mathbf{F}^\mathrm{h})^\mathrm{T}\bar{\mathbf{g}}\mathbf{F}^\mathrm{h}$ and $\mathbf{C}^\mathrm{v} = (\mathbf{F}^\mathrm{v})^\mathrm{T}\bar{\mathbf{g}}\mathbf{F}^\mathrm{v}$. The Cauchy–Green strain tensor, $\mathbf{E}$, given by

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{g}),$$

where

$$\mathbf{g} = \mathbf{g}^\mathrm{h} + \mathbf{g}^\mathrm{v}, \qquad (\mathbf{g}^\mathrm{h})_{ij} = (\mathbf{g}^\mathrm{v})_{ij} = g_{ij}$$

has the property of vanishing in the reference configuration.

Using definitions (2.27) and (2.28), we obtain the following representation for strain measures $\mathbf{C}^\mathrm{h}$ and $\mathbf{C}^\mathrm{v}$,

$$\begin{aligned}\mathbf{C}^\mathrm{h} = \Big(&\partial_i X^a \partial_l X^b + \bar{\partial}_m X^b \bar{\partial}_l G^m \bar{\partial}_k X^a \bar{\partial}_i G^k + {}^\mathrm{h}\Gamma^b_{cl}\,{}^\mathrm{h}\Gamma^a_{di} X^c X^d \\ &- \partial_{(l} X^{(b} \bar{\partial}_{|k|} X^{a)} \bar{\partial}_{i)} G^k - \bar{\partial}_m X^{(b} \bar{\partial}_{(l} G^{|m|\mathrm{h}} \Gamma^{a)}_{|c|i)} X^c + \partial_{(l} X^{(b\mathrm{h}} \Gamma^{a)}_{|c|i)} X^c \Big) \bar{g}_{ab} \mathbf{g}^l \otimes \mathbf{g}^i, \\ \mathbf{C}^\mathrm{v} = \Big(&\bar{\partial}_i X^a \bar{\partial}_l X^b + \bar{\partial}_{(l} X^{(b\mathrm{v}} \Gamma^{a)}_{|c|i)} X^c + {}^\mathrm{v}\Gamma^a_{di}\,{}^\mathrm{v}\Gamma^b_{el} X^d X^e \Big) \bar{g}_{ab}\,{}^\mathrm{v}\mathbf{z}^l \otimes {}^\mathrm{v}\mathbf{z}^i,\end{aligned} \tag{2.35}$$

where ( ) means the symmetric part with respect to the enclosed indices, and the sign | | around the index is used to exclude it from the symmetrization operation. The interrelated pair of measures, Eq. (2.35), defined in the invariant manner, is strictly connected with the analytical form of the functional L characterizing the local topography of deformation process (cf. Kocks et al., 1975). Using the introduced strain measures one can, in principle, completely describe any state or distortion of the three-dimensional body in terms of some suitable distribution and interaction energies of dislocations and/or other material defects.

Before concluding this section, one special case may be noted. In the case of the classical continuum mechanics, when the internal vector field $\mathbf{y}$ is neglected, then $\mathbf{C}^{\mathrm{h}}$ has the classical sense, while $\mathbf{C}^{\mathrm{v}}$ is singular if $\mathbf{X}$ and $\mathbf{g}$ are functions of $\mathbf{x}$ alone. In the other limiting case, when a residual state inside the body is non-singular both $\mathbf{C}^{\mathrm{h}}$ and $\mathbf{C}^{\mathrm{v}}$ have to be considered. To specify the connection coefficients ${}^{\mathrm{v}}\Gamma^{a}_{bk}, \bar{\partial}_j G^i$ and ${}^{\mathrm{h}}\Gamma^{a}_{bk}$, we have to estimate the local dislocated (distorted) state of $\mathscr{B}$ by considering its fundamental functional L. In general, $\mathbf{X} = \mathbf{X}(\mathbf{x}, \mathbf{y})$ and we are free to choose either $\mathbf{C} = \mathbf{C}^{\mathrm{h}} + \mathbf{C}^{\mathrm{v}}$ defined on the Finslerian bundle, or $\mathbf{C}^{\mathrm{v}}$ defined on the fibre space, or $\mathbf{C}^{\mathrm{h}}$ defined on the base space of the Finslerian bundle.

3. Variational formulation

In this section, our purpose is to provide a complete set of balance laws and boundary conditions for macro- and microstresses of Newtonian and Eshelbian type for a dissipative model of oriented continuum with microstructure, say, with inhomogeneities and evolving defects. We extend and generalize the variational formulation technique developed and applied by Rund (1966), Lovelock and Rund (1975), Naghdi and Srinivasa (1993a), Saczuk (1993, 1996), Stumpf and Le (1990, 1992), Maugin and Trimarco (1992), Maugin (1993) and Stumpf and Saczuk (2000). Concerning the transversality conditions, we will take the line adopted by Edelen (1981) and Saczuk (1993).

The direct problem of the calculus of variation is concerned with finding local sections (fields) of $\mathscr{L}_t$ which give critical (equilibrium) points of the integral $\int \mathscr{L}_t \, \mathrm{d}V \, \mathrm{d}t$. This problem is related to the problem of finding solutions of the Euler–Lagrange equations.

3.1. The first-order action integral

According to the defined macro- and microdeformation gradients (2.26), the first-order functional is defined as

$$I_t = \int_G \int_T \mathscr{L}_t(\mathbf{x}, \mathbf{y}, t, \mathbf{X}, \mathbf{F}^{\mathrm{h}}, \mathbf{F}^{\mathrm{v}}, \dot{\mathbf{X}}^{\mathrm{h}}, \dot{\mathbf{X}}^{\mathrm{v}}) \, \mathrm{d}V \, \mathrm{d}t, \tag{3.1}$$

where $\dot{\mathbf{X}}^{\mathrm{h}}$ and $\dot{\mathbf{X}}^{\mathrm{v}}$ are the time of derivative of Eq. (2.1) reduced to the macro- and microspace, respectively (cf. Eq. (3.14)). We therefore assume that the Lagrangian density functional $\mathscr{L}_t$, described by the first-order derivatives of state variables, is a smooth map

$$\mathscr{L}_t : \mathbb{E}^6 \times \mathbb{R} \times J^1(\mathbb{E}^7) \to \mathbb{R},$$

where $J^1(\cdot)$ is the first jet bundle (cf. Libermann and Marle, 1986). We also assume that the Lagrangian is invariant under arbitrary transformations of coordinates x^i and y^i with the non-singular Jacobian. A common assumption splitting the functional $\mathscr{L}_t$ into potential and kinetic parts is not introduced here. Moreover, G denotes a fixed, closed and simply-connected region (a compact six-dimensional manifold) in the six-dimensional space of $(\mathbf{x}, \mathbf{y})$, bounded by the surface ∂G and T a time interval. The region G is here identified with a part of the body $\mathscr{B}$. The volume element associated with any of the inelastically distorted states considered in Eq. (3.1) is defined by

$$\mathrm{d}V = \sqrt{g}\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} = \sqrt{g}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3, \tag{3.2}$$

where g is the determinant of the metric tensor $\mathbf{g} = \mathbf{g}^{\mathrm{h}} \oplus \mathbf{g}^{\mathrm{v}}$ with $\mathbf{g}^{\mathrm{h}}$ and $\mathbf{g}^{\mathrm{v}}$ defined as

$$\mathbf{g}^{\mathrm{h}} = g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \qquad \mathbf{g}^{\mathrm{v}} = g_{ij}{}^{\mathrm{v}}\mathbf{g}^i \otimes {}^{\mathrm{v}}\mathbf{g}^j.$$

The variational derivative of functional (3.1) yields

$$\delta I_t = \int_G \int_T (\delta \mathscr{L}_t) \mathrm{d}V\,\mathrm{d}t + \int_G \int_T \mathscr{L}_t \,\delta(\mathrm{d}V\,\mathrm{d}t), \tag{3.3}$$

where

$$\delta \mathscr{L}_t = \nabla^{\mathrm{h}} \mathscr{L}_t \cdot \delta \mathbf{x} + \nabla^{\mathrm{v}} \mathscr{L}_t \cdot \delta \mathbf{y} + \frac{\partial \mathscr{L}_t}{\partial \mathbf{X}^{\mathrm{h}}} \cdot \delta \mathbf{X}^{\mathrm{h}} + \frac{\partial \mathscr{L}_t}{\partial \mathbf{X}^{\mathrm{v}}} \cdot \delta \mathbf{X}^{\mathrm{v}} + \frac{\partial \mathscr{L}_t}{\partial \mathbf{F}^{\mathrm{h}}} \cdot \delta \mathbf{F}^{\mathrm{h}} + \frac{\partial \mathscr{L}_t}{\partial \mathbf{F}^{\mathrm{v}}} \cdot \delta \mathbf{F}^{\mathrm{v}} + \frac{\partial \mathscr{L}_t}{\partial \dot{\mathbf{X}}^{\mathrm{h}}} \cdot \delta \dot{\mathbf{X}}^{\mathrm{h}} + \frac{\partial \mathscr{L}_{tv}}{\partial \dot{\mathbf{X}}} \cdot \delta \dot{\mathbf{X}}^{\mathrm{v}} \tag{3.4}$$

and, using Eq. (3.2) and the assumption $\delta t = \mathrm{d}t$,

$$\delta(\mathrm{d}V \mathrm{d}t) = \delta(\sqrt{g}\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}t) = [D^{\mathrm{h}}(\delta \mathbf{x}) + D^{\mathrm{v}}(\delta \mathbf{y})]\mathrm{d}V\mathrm{d}t, \tag{3.5}$$

with

$$D^{\mathrm{h}}(\delta \mathbf{x}) = \nabla^{\mathrm{h}}(\delta \mathbf{x}) + \frac{\partial(\delta \mathbf{x})}{\partial \mathbf{X}} \nabla^{\mathrm{h}} \mathbf{X}, \quad D^{\mathrm{v}}(\delta \mathbf{y}) = \nabla^{\mathrm{v}}(\delta \mathbf{y}) + \frac{\partial(\delta \mathbf{y})}{\partial \mathbf{X}} \nabla^{\mathrm{v}} \mathbf{X}.$$

All variational derivatives are obtained [6] under the assumption that the system (the body with loads) admits a one-parameter transformation group acting on the independent and dependent variables in the form

$$\begin{aligned} \bar{x}^i &= x^i + v_x^i(x^m, y^m, t, X^b)\epsilon + \mathrm{o}(\epsilon), \\ \bar{y}^i &= y^i + v_y^i(x^m, y^m, t, X^b)\epsilon + \mathrm{o}(\epsilon), \\ \bar{X}^a &= X^a + v_X^a(x^m, y^m, t, X^b)\epsilon + \mathrm{o}(\epsilon), \end{aligned} \tag{3.6}$$

where ϵ denotes a scalar parameter, while $v_x^i(\cdot), v_y^i(\cdot)$ and $v_X^a(\cdot)$ are class C^1 functions of their variables such that

$$\bar{x}^i(0) = x^i, \quad \bar{y}^i(0) = y^i, \quad \bar{X}^a(0) = X^a \tag{3.7}$$

for $\epsilon \to 0$.

To obtain the mechanical version of the balance laws for $\mathscr{B}$ we define macro- and micromomentum vectors

$$\mathbf{p}^{\mathrm{h}} = \frac{\partial \mathscr{L}_t}{\partial \dot{\mathbf{X}}^{\mathrm{h}}}, \qquad \mathbf{p}^{\mathrm{v}} = \frac{\partial \mathscr{L}_t}{\partial \dot{\mathbf{X}}^{\mathrm{v}}}, \tag{3.8a, b}$$

configurational macro- and micromomentum vectors

$$\mathbb{p}^{\mathrm{h}} = \frac{\partial \mathscr{L}_t}{\partial \dot{\mathbf{x}}}, \qquad \mathbb{p}^{\mathrm{v}} = \frac{\partial \mathscr{L}_t}{\partial \dot{\mathbf{y}}}, \tag{3.9}$$

macro- and microstress tensors

$$\mathbf{T}^{\mathrm{h}} = -\frac{\partial \mathscr{L}_t}{\partial \mathbf{F}^{\mathrm{h}}}, \qquad \mathbf{T}^{\mathrm{v}} = -\frac{\partial \mathscr{L}_t}{\partial \mathbf{F}^{\mathrm{v}}}, \tag{3.10}$$

configurational macro- and microstress tensors

$$\mathbb{T}^{\mathrm{h}} = -\mathscr{L}_t \mathbf{1}^{\mathrm{h}} - (\mathbf{F}^{\mathrm{h}})^{\mathrm{T}} \mathbf{T}^{\mathrm{h}}, \qquad \mathbb{T}^{\mathrm{v}} = -\mathscr{L}_t \mathbf{1}^{\mathrm{v}} - (\mathbf{F}^{\mathrm{v}})^{\mathrm{T}} \mathbf{T}^{\mathrm{v}}, \tag{3.11}$$

[6] Existence and continuity of derivatives will be assumed without explicit mention.

external and internal body forces

$$\mathbf{f}^{\mathrm{h}} = \frac{\partial \mathscr{L}_t}{\partial \mathbf{X}^{\mathrm{h}}}, \qquad \mathbf{f}^{\mathrm{v}} = \frac{\partial \mathscr{L}_t}{\partial \mathbf{X}^{\mathrm{v}}}, \tag{3.12a, b}$$

and macro- and microinhomogeneity forces

$$\mathbb{f}^{\mathrm{h}} = \nabla^{\mathrm{h}} \mathscr{L}_t, \qquad \mathbb{f}^{\mathrm{v}} = \nabla^{\mathrm{v}} \mathscr{L}_t. \tag{3.13}$$

At a corner point $\mathbf{X} = (\mathbf{X}^{\mathrm{h}}, \mathbf{X}^{\mathrm{v}})$ the above derivatives are to be interpreted as left- or right-hand derivatives. Moreover, in Eqs. (3.8a,b) and (3.9), we have adapted the following definitions of time-dependent objects:

$$\begin{aligned} \dot{\mathbf{X}}^{\mathrm{h}} &\equiv \mathbf{v}^{\mathrm{h}} = (\partial_t \boldsymbol{\chi})^{\mathrm{h}}, \quad \dot{\mathbf{X}}^{\mathrm{v}} \equiv \mathbf{v}^{\mathrm{v}} = (\partial_t \boldsymbol{\chi})^{\mathrm{v}}, \\ \dot{\mathbf{x}}^{\mathrm{h}} &\equiv \mathbb{v}^{\mathrm{h}} = (\partial_t \boldsymbol{\chi}^{-1})^{\mathrm{h}}, \quad \dot{\mathbf{y}}^{\mathrm{v}} \equiv \mathbb{v}^{\mathrm{v}} = (\partial_t \boldsymbol{\chi}^{-1})^{\mathrm{v}} \end{aligned} \tag{3.14}$$

calculated for a fixed particle of $\mathscr{B}$, with ∂_t denoting the partial derivative with respect to time t. One should stress here that the decomposition of the cited objects into $(\ )^{\mathrm{h}}$ and $(\ )^{\mathrm{v}}$ parts is unique. Such a notation is used extensively throughout the remaining sections together with the convention

$$\delta \mathbf{X}^{\mathrm{h}} = (\delta \mathbf{X})^{\mathrm{h}}, \qquad \delta \mathbf{X}^{\mathrm{v}} = (\delta \mathbf{X})^{\mathrm{v}}.$$

Introducing Eqs. (3.4), (3.5) and (3.8a,b)–(3.13) into Eq. (3.3) and using the divergence theorem, we obtain

$$\begin{aligned} \delta I_t = &\int_G \int_T [(-\dot{\mathbf{p}}^{\mathrm{h}} + \mathbf{f}^{\mathrm{h}} + \operatorname{Div} \mathbf{T}^{\mathrm{h}}) \cdot \delta \mathbf{X}^{\mathrm{h}} + (-\dot{\mathbf{p}}^{\mathrm{v}} + \mathbf{f}^{\mathrm{v}} + \operatorname{Div} \mathbf{T}^{\mathrm{v}}) \cdot \delta \mathbf{X}^{\mathrm{v}} \\ &+ (-\dot{\mathbb{p}}^{\mathrm{h}} + \mathbb{f}^{\mathrm{h}} + \operatorname{Div} \mathbb{T}^{\mathrm{h}}) \cdot \delta \mathbf{x} + (-\dot{\mathbb{p}}^{\mathrm{v}} + \mathbb{f}^{\mathrm{v}} + \operatorname{Div} \mathbb{T}^{\mathrm{v}}) \cdot \delta \mathbf{y}] \mathrm{d}V \,\mathrm{d}t \\ &- \int_{\partial G} \int_T (\mathbb{T}^{\mathrm{h}} \mathbf{n}^{\mathrm{h}} \cdot \delta \mathbf{x} + \mathbf{T}^{\mathrm{h}} \mathbf{n}^{\mathrm{h}} \cdot \delta \mathbf{X}^{\mathrm{h}} + \mathbb{T}^{\mathrm{v}} \mathbf{n}^{\mathrm{v}} \cdot \delta \mathbf{y} + \mathbf{T}^{\mathrm{v}} \mathbf{n}^{\mathrm{v}} \cdot \delta \mathbf{X}^{\mathrm{v}}) \mathrm{d}S \,\mathrm{d}t, \end{aligned} \tag{3.15}$$

where $\mathrm{d}S$ denotes the element of area of the hypersurface ∂G bounding G, $\mathbf{n}^{\mathrm{h}}$ and $\mathbf{n}^{\mathrm{v}}$ are the suitable orientated unit normal vectors to ∂G at the macro- and microlevel and

$$\operatorname{Div} \mathbf{T}^{\mathrm{h}} = D^{\mathrm{h}} \mathbf{T}^{\mathrm{h}} - \bar{\partial} \mathbf{G} D^{\mathrm{v}} \mathbf{T}^{\mathrm{h}} - \mathbf{T}^{\mathrm{h}} \boldsymbol{\Gamma}^{\mathrm{h}}, \quad \operatorname{Div} \mathbf{T}^{\mathrm{v}} = D^{\mathrm{v}} \mathbf{T}^{\mathrm{v}} - \mathbf{T}^{\mathrm{v}} \boldsymbol{\Gamma}^{\mathrm{v}},$$

$$\operatorname{Div} \mathbb{T}^{\mathrm{h}} = -\nabla^{\mathrm{h}} \mathscr{L}_t - (\mathbf{F}^{\mathrm{h}})^{\mathrm{T}} (\mathbf{f}^{\mathrm{h}} + \operatorname{Div} \mathbf{T}^{\mathrm{h}}), \quad \operatorname{Div} \mathbb{T}^{\mathrm{v}} = -\nabla^{\mathrm{v}} \mathscr{L}_t - (\mathbf{F}^{\mathrm{v}})^{\mathrm{T}} (\mathbf{f}^{\mathrm{v}} + \operatorname{Div} \mathbf{T}^{\mathrm{v}})$$

the generalized divergence operator Div of $\mathbf{T}^{\mathrm{h}}, \mathbf{T}^{\mathrm{v}}, \mathbb{T}^{\mathrm{h}}$ and $\mathbb{T}^{\mathrm{v}}$.

The variation (3.15) was obtained under the following initial conditions:

$$\begin{aligned} \mathbf{p}^{\mathrm{h}}\big|_0 &= \mathbf{p}^{\mathrm{h}}\big|_t = \mathbf{0}, \qquad \mathbf{p}^{\mathrm{v}}\big|_0 = \mathbf{p}^{\mathrm{v}}\big|_t = \mathbf{0}, \\ \mathbb{p}^{\mathrm{h}}\big|_0 &= \mathbb{p}^{\mathrm{h}}\big|_t = \mathbf{0}, \qquad \mathbb{p}^{\mathrm{v}}\big|_0 = \mathbb{p}^{\mathrm{v}}\big|_t = \mathbf{0} \end{aligned} \tag{3.16}$$

at the initial and final times. The stationary conditions for an arbitrary time variation δt,

$$\mathbf{v}^{\mathrm{h}} \cdot \dot{\mathbf{p}}^{\mathrm{h}} + \mathbb{v}^{\mathrm{h}} \cdot \dot{\mathbb{p}}^{\mathrm{h}} = 0, \qquad \mathbf{v}^{\mathrm{v}} \cdot \dot{\mathbf{p}} \mathrm{v} + \mathbb{v}^{\mathrm{v}} \cdot \dot{\mathbb{p}}^{\mathrm{v}} = 0$$

represent kinematical compatibility conditions between physical $\mathbf{v}^{\mathrm{h}}(\mathbf{v}^{\mathrm{v}})$ and material $\mathbb{v}^{\mathrm{h}}(\mathbb{v}^{\mathrm{v}})$ velocities and physical $\dot{\mathbf{p}}^{\mathrm{h}}(\dot{\mathbf{p}}^{\mathrm{v}})$ and material $\dot{\mathbb{p}}^{\mathrm{h}}(\dot{\mathbb{p}}^{\mathrm{v}})$ momentum rates at both levels.

The constitutive relations (3.8a,b)–(3.13) describing the conservative part of the model may be deduced from the Clausius–Duhem inequality.

The starting point for the investigation of the dissipative model of oriented continuum with microstructure is the action integral (3.15) modified by including into its integrand prescribed tractions $\mathbf{t}^{\mathrm{h}}, \mathbf{t}^{\mathrm{v}}$, whose components are taken in the actual configuration and configurational boundary stresses $\mathbb{t}^{\mathrm{h}}, \mathbb{t}^{\mathrm{v}}$, whose components are taken in the reference configuration on each part of ∂G. We assume that the action I_t associated with the motion of $\mathscr{B}$ satisfies the relation

$$\delta I_t = -\int_G \int_T (\mathbf{t}^{\mathrm{h}} \cdot \delta \mathbf{X}^{\mathrm{h}} + \mathbf{t}^{\mathrm{v}} \cdot \delta \mathbf{X}^{\mathrm{v}} + \mathbb{t}^{\mathrm{h}} \cdot \delta \mathbf{x} + \mathbb{t}^{\mathrm{v}} \cdot \delta \mathbf{y})\, \mathrm{d}V\, \mathrm{d}t. \tag{3.17}$$

It is also assumed that body forces $\mathbf{f}^{\mathrm{h}}$ and $\mathbf{f}^{\mathrm{v}}$ acting on each volume element $\mathrm{d}V$ of $\mathscr{B}$ are defined by Eq. (3.12a,b), and, analogously, configurational body forces $\mathbb{f}^{\mathrm{h}}$ and $\mathbb{f}^{\mathrm{v}}$ acting on each volume $\mathrm{d}V$ of $\mathscr{B}$ are defined by Eq. (3.13).

3.2. Balance laws, boundary and transversality conditions

The dynamical laws and boundary conditions for deformational and configurational forces resulting from the stationarity condition (3.15) together with Eq. (3.17) are following:

(a) The balance of deformational and configurational macromomentum

$$\dot{\mathbf{p}}^{\mathrm{h}} = \mathbf{f}^{\mathrm{h}} + \mathrm{Div}\,\mathbf{T}^{\mathrm{h}}, \qquad \dot{\mathbb{p}}^{\mathrm{h}} = \mathbb{f}^{\mathrm{h}} + \mathrm{Div}\,\mathbb{T}^{\mathrm{h}}, \tag{3.18}$$

where $\mathbf{p}^{\mathrm{h}}$ is the momentum vector, $\mathbb{p}^{\mathrm{h}}$ the Eshelbian momentum vector, $\mathbf{T}^{\mathrm{h}}$ the first Piola–Kirchhoff macrostress tensor, $\mathbb{T}^{\mathrm{h}}$ the Eshelbian macrostress tensor, $\mathbf{f}^{\mathrm{h}}$ the external macrobody force and $\mathbb{f}^{\mathrm{h}}$ the material macroinhomogenity force.

(b) The balance of moment of deformational and configurational macromomentum

$$\mathbf{F}^{\mathrm{h}}(\mathbf{T}^{\mathrm{h}})^{\mathrm{T}} = \mathbf{T}^{\mathrm{h}}(\mathbf{F}^{\mathrm{h}})^{\mathrm{T}}, \qquad \mathbf{C}^{\mathrm{h}}(\mathbb{T}^{\mathrm{h}})^{\mathrm{T}} = \mathbb{T}^{\mathrm{h}}\mathbf{C}^{\mathrm{h}}. \tag{3.19}$$

(c) The balance of deformational and configurational micromomentum

$$\dot{\mathbf{p}}^{\mathrm{v}} = \mathbf{f}^{\mathrm{v}} + \mathrm{Div}\,\mathbf{T}^{\mathrm{v}}, \qquad \dot{\mathbb{p}}^{\mathrm{v}} = \mathbb{f}^{\mathrm{v}} + \mathrm{Div}\,\mathbb{T}^{\mathrm{v}}, \tag{3.20}$$

where $\mathbf{p}^{\mathrm{v}}$ is the micromomentum vector, $\mathbb{p}^{\mathrm{v}}$ the Eshelbian micromomentum vector, $\mathbf{T}^{\mathrm{v}}$ the first Piola–Kirchhoff microstress tensor, $\mathbb{T}^{\mathrm{v}}$ the Eshelbian microstress tensor, $\mathbf{f}^{\mathrm{v}}$ the internal microbody force and $\mathbb{f}^{\mathrm{v}}$ the material microinhomogeneity force.

(d) The balance of moment of deformational and configurational macromomentum

$$\mathbf{F}^{\mathrm{v}}(\mathbf{T}^{\mathrm{v}})^{\mathrm{T}} = \mathbf{T}^{\mathrm{v}}(\mathbf{F}^{\mathrm{v}})^{\mathrm{T}}, \qquad \mathbf{C}^{\mathrm{v}}(\mathbb{T}^{\mathrm{v}})^{\mathrm{T}} = \mathbb{T}^{\mathrm{v}}\mathbf{C}^{\mathrm{v}}. \tag{3.21}$$

(e) The traction boundary conditions and the configurational traction boundary conditions

$$\mathbf{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}} = \mathbf{t}^{\mathrm{h}}, \quad \mathbf{T}^{\mathrm{v}}\mathbf{n}^{\mathrm{v}} = \mathbf{t}^{\mathrm{v}}, \quad \mathbb{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}} = \mathbb{t}^{\mathrm{h}}, \quad \mathbb{T}^{\mathrm{v}}\mathbf{n}^{\mathrm{v}} = \mathbb{t}^{\mathrm{v}} \tag{3.22}$$

where $\mathbf{n}^{\mathrm{h}}$ and $\mathbf{n}^{\mathrm{v}}$ are the outer normal vectors to the boundary $\partial\mathscr{B}$ at macro- and microlevel, respectively.

(f) The transversality conditions

$$-\mathbb{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}} \cdot \delta\mathbf{x} = \mathbf{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}} \cdot \delta\mathbf{X}^{\mathrm{h}}, \qquad -\mathbb{T}^{\mathrm{v}}\mathbf{n}^{\mathrm{v}} \cdot \delta\mathbf{y} = \mathbf{T}^{\mathrm{v}}\mathbf{n}^{\mathrm{v}} \cdot \delta\mathbf{X}^{\mathrm{v}} \tag{3.23}$$

are the result of demanding that the variational identity (3.15) is equal to zero, $\delta I_t = 0$, for all variations $(\delta\mathbf{x}, \delta\mathbf{y}, \delta\mathbf{X}^{\mathrm{h}}, \delta\mathbf{X}^{\mathrm{v}})$. A set of variations $(\delta\mathbf{x}, \delta\mathbf{y}, \delta\mathbf{X}^{\mathrm{h}}, \delta\mathbf{X}^{\mathrm{v}})$ satisfies the transversality conditions on ∂G if and only if the conditions (3.23) are satisfied at all points of ∂G (Edelen, 1981; Saczuk, 1993).

One has to note that the transversality conditions must be satisfied, if the action of the functional I_t has to be stationary. An integration of both sides of Eq. (3.23) over an arbitrary boundary Σ of G, say a new fracture surface, leads to the virtual surface principles:

$$-\int_\Sigma \mathbb{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}} \cdot \delta\mathbf{x}\,\mathrm{d}\Sigma = \int_\Sigma \mathbf{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}} \cdot \delta\mathbf{X}^{\mathrm{h}}\,\mathrm{d}\Sigma, \tag{3.24}$$

$$-\int_{\Sigma} \mathbb{T}^{\mathrm{v}} \mathbf{n}^{\mathrm{v}} \cdot \delta \mathbf{y} \mathrm{d}\Sigma = \int_{\Sigma} \mathbf{T}^{\mathrm{v}} \mathbf{n}^{\mathrm{v}} \cdot \delta \mathbf{X}^{\mathrm{v}} \mathrm{d}\Sigma. \tag{3.25}$$

Here, Σ may denote either an arbitrary part ∂G of the body $\mathscr{B}$ or a certain internal boundary within the body like that connected with the evolution of new crack surfaces in the fracture process. The principles (3.24) and (3.25) define an equality between generalized field forces on variable boundaries and the actual forces that act directly on the boundaries in their motion.

3.3. The dissipation inequality

The second law of thermodynamics is the assertion that, when thermal effects are neglected, the rate of energy increase cannot exceed the total expended power. Written in the form of the Clausius–Duhem inequality (or the entropy production inequality) it is identified here with the sufficiency condition for the functional (3.1) (cf. Saczuk, 1997b; Stumpf and Saczuk, 2000).

The Euler–Lagrange equations (3.18)–(3.21) of Eq. (3.1) are not, in general, sufficient for the functional I_t to assume the extreme value. The sufficiency conditions for I_t, strictly connected with the convexity conditions demanded by the dissipation inequality, can be easily obtained within the so-called method of equivalent integrals (Rund, 1966; Lovelock and Rund, 1975). This method, in principle, requires the construction of a function Λ_t (a counterpart of the total derivative) defined on Eq. (2.1). This function being independent of the choice of the subspace (2.1) is the integrand $\tilde{\mathscr{L}}_t(\mathbf{x},\mathbf{y}) = \mathscr{L}_t(\mathbf{x},\mathbf{y}) - \Lambda_t(\mathbf{x},\mathbf{y})$ of a new action integral

$$\tilde{I}_t(\mathbf{x},\mathbf{y}) = \int_G \tilde{\mathscr{L}}_t(\mathbf{x},\mathbf{y})\mathrm{d}V$$

which, by definition, provides an extreme value to the same solutions as the solutions of the original problem defined by $I_t = \int_G \mathscr{L}_t \mathrm{d}V$.

Within the cited method, we have to construct a function Λ_t defined on

$$X^i = X^i(\mathbf{x},\mathbf{y},t), \qquad \theta = \theta(\mathbf{x},\mathbf{y},t). \tag{3.26}$$

(Here we assume that $\mathscr{L}_t$ is dependent also on temperature θ and its gradients $\nabla^{\mathrm{h}}\theta$ and $\nabla^{\mathrm{v}}\theta$.) The function Λ_t, which gives rise to the equivalent variational problem, can be given as follows (cf. Rund, 1966; Stumpf and Saczuk, 2000):

$$\begin{aligned}\Lambda_t(\tilde{\mathbf{F}}^{\mathrm{h}}, \tilde{\mathbf{F}}^{\mathrm{v}}, \nabla^{\mathrm{h}}\tilde{\theta}, \nabla^{\mathrm{v}}\tilde{\theta}) = \mathscr{L}_t^{-5} \det \Bigg[& \mathscr{L}_t \mathbf{g} \oplus \mathbf{g} + \left(\frac{\partial \mathscr{L}_t}{\partial \mathbf{F}^{\mathrm{h}}} + \frac{\partial \mathscr{L}_t}{\partial \mathbf{F}^{\mathrm{v}}}\right)(\tilde{\mathbf{F}}^{\mathrm{h}} - \mathbf{F}^{\mathrm{h}} + \tilde{\mathbf{F}}^{\mathrm{v}} - \mathbf{F}^{\mathrm{v}}) \\ & + \left(\frac{\partial \mathscr{L}_t}{\partial \nabla^{\mathrm{h}}\theta} + \frac{\partial \mathscr{L}_t}{\partial \nabla^{\mathrm{v}}\theta}\right) \otimes (\nabla^{\mathrm{h}}\tilde{\theta} - \nabla^{\mathrm{h}}\theta + \nabla^{\mathrm{v}}\tilde{\theta} - \nabla^{\mathrm{v}}\theta)\Bigg],\end{aligned} \tag{3.27}$$

where quantities $\nabla^{\mathrm{h}}\theta$ and $\nabla^{\mathrm{v}}\theta$ represent two temperature gradients, first, the temperature gradient of material points, second, the internal temperature gradient of the internal structure of the point. [7] Moreover, the arguments $\mathbf{F}^{\mathrm{h}}, \mathbf{F}^{\mathrm{v}}$ and $\nabla^{\mathrm{h}}\theta, \nabla^{\mathrm{v}}\theta$ refer here to the geodesic field in the analysed subspace, while $\tilde{\mathbf{F}}^{\mathrm{h}}, \tilde{\mathbf{F}}^{\mathrm{v}}$ and $\nabla^{\mathrm{h}}\tilde{\theta}, \nabla^{\mathrm{v}}\tilde{\theta}$ are arbitrary.

Under this preparation, the sufficiency condition of Weierstrass for thermo-inelastic process of $\mathscr{B}$ has the form

$$\mathscr{E} = \mathscr{E}(\mathbf{F}^{\mathrm{h}}, \mathbf{F}^{\mathrm{v}}, \nabla^{\mathrm{h}}\theta, \nabla^{\mathrm{v}}\theta, \tilde{\mathbf{F}}^{\mathrm{h}}, \tilde{\mathbf{F}}^{\mathrm{v}}, \nabla^{\mathrm{h}}\tilde{\theta}, \nabla^{\mathrm{v}}\tilde{\theta}) \geqslant 0, \tag{3.28}$$

[7] The multiplier $\mathscr{L}_t^{-5}$ in Eq. (3.27) is a consequence of the dimension of $\mathscr{B}$ and definition of the determinant.

valid for all $\tilde{\mathbf{F}}^h = \mathbf{Q}^h\mathbf{F}^h$ and $\tilde{\mathbf{F}}^v = \mathbf{Q}^v\mathbf{F}^v$ with arbitrary positive-definite tensors $\mathbf{Q}^h$ and $\mathbf{Q}^v$ and for all constants ϑ such that $\tilde{\theta} = \theta + \vartheta > 0$. The Weierstrass excess function $\mathscr{E}$ is then defined by

$$\mathscr{E} = \mathscr{L}_t(\tilde{\mathbf{F}}^h, \tilde{\mathbf{F}}^v, \nabla^h\tilde{\theta}, \nabla^v\tilde{\theta}) - \Lambda_t(\tilde{\mathbf{F}}^h, \tilde{\mathbf{F}}^v, \nabla^h\tilde{\theta}, \nabla^v\tilde{\theta}). \tag{3.29}$$

Identification of the corresponding differences in Eq. (3.27) with their increments (thermodynamic forces) leads to a local Clausius–Duhem type inequality accounting for thermal effects in the form

$$\rho_0\dot{\mathscr{L}}_r - (\mathbf{T}^h + \mathbf{T}^v)\cdot(\dot{\mathbf{F}}^h + \dot{\mathbf{F}}^v) + \theta^{-1}(\nabla^h\theta + \nabla^v\theta)\cdot(\mathbf{H}^h + \mathbf{H}^v) \leqslant 0, \tag{3.30}$$

where $\dot{\mathscr{L}}_r(\mathscr{L}_t = \rho_0\mathscr{L}_r)$ is the rate of energy functional per unit mass, ρ_0 the mass density in the reference configuration, η the entropy, θ the absolute temperature, $\mathbf{H}^h$ and $\mathbf{H}^v$ macro- and microheat flux vectors. Here, the thermal terms were obtained within the theory of hyperbolic heat transfer (Gurtin and Pipkin, 1968), where a thermal path $(\theta(t), \nabla^h\theta(t), \nabla^v\theta(t))$ is identified with its summed history $(\bar{\theta}^t(s), \nabla^h\bar{\theta}^t(s), \nabla^v\bar{\theta}^t(s))$, whose rates are given by

$$\frac{d}{dt}\bar{\theta}^t(s) = \theta(t) - \theta^t(s), \qquad \frac{d}{dt}\nabla^h\bar{\theta}^t(s) = \nabla^h\theta(t) - \nabla^h\theta^t(s),$$

$$\frac{d}{dt}\nabla^v\bar{\theta}^t(s) = \nabla^v\theta(t) - \nabla^v\theta^t(s).$$

In the case of a pure mechanical model, this inequality reduces to (cf. Hanyga, 1990)

$$\rho_0\dot{\mathscr{L}}_r - (\mathbf{T}^h + \mathbf{T}^v)\cdot(\dot{\mathbf{F}}^h + \dot{\mathbf{F}}^v) \leqslant 0.$$

From the fact that the right-hand side of Eq. (3.30) never exceeds some finite upper bound, equal $\rho_0\theta\dot{\eta}$ (Day, 1972), the inequality (3.30) leads to the final form

$$\rho_0(\eta\dot{\theta} + \dot{\Psi}) - (\mathbf{T}^h + \mathbf{T}^v)\cdot(\dot{\mathbf{F}}^h + \dot{\mathbf{F}}^v) + \theta^{-1}(\nabla^h\theta + \nabla^v\theta)\cdot(\mathbf{H}^h + \mathbf{H}^v) \leqslant 0 \tag{3.31}$$

expressed in terms of the Helmholtz free energy $\Psi = \mathscr{L}_r - \theta\eta$.

In terms of the entropy production, the dissipation inequality (3.31) can be written as

$$\sigma = \sigma_{\text{int}} + \sigma_{\text{th}} \geqslant 0, \tag{3.32}$$

where the internal coupled dissipation σ_{int} is defined by

$$\sigma_{\text{int}} = \mathbf{S}^v\cdot\dot{\mathbf{E}}^h + \mathbf{S}^h\cdot\dot{\mathbf{E}}^v, \tag{3.33}$$

and the thermal dissipation σ_{th} by

$$\sigma_{\text{th}} = -\theta^{-1}(\nabla^h\theta + \nabla^v\theta)\cdot(\mathbf{H}^h + \mathbf{H}^v). \tag{3.34}$$

If Eq. (3.32) holds with the equality sign, the thermodynamic process is called reversible, otherwise irreversible.

4. Applications

The theory developed in Sections 2 and 3 is based mainly on the concept of Finslerian bundle formulated on the tangent space to the original base manifold (Matsumoto, 1986). A certain geometrical aspects of the Finslerian geometry can be formulated in terms of the lifting of geometric object from the base manifold to its tangent bundle (Yano and Ishihara, 1973). The aim of this theory, which is presented in Section 4.1 only in outline, is to obtain the higher-order geometries of the base manifold. For simplicity, our consideration presented below is devoted to generalize a classical damage tensor to the one defined on the tangent bundle.

In Section 4.2, we formulate the macro–micro constitutive equations and the associated phenomenological macro constitutive relations for the thermo-inelastic processes for the Newtonian part of the direct motion. This leads to a close correlation with a continuum mechanics constitutive modelling.

The possibility to decouple macro- and micromotions into components of Euclidean space structure was a guiding idea to take into consideration mainly the contributions of the macromotion to propose in Section 4.3 a strain-induced crack propagation criterion, defined by the difference between the strain energy release rate and the rate of the surface energy of the crack. The used simplification turns out to be reasonable, since our manifold-theoretic setting yields an appropriate averaged internal structure representation.

4.1. A classical damage concept on the tangent bundle

In this section we investigate, based on the theory of lifting of scalar, vector and tensor fields from the base manifold to its tangent bundle (Yano and Ishihara, 1973; Saczuk, 1992, 1994), the damage deformation gradient on the tangent bundle over the damaged medium with an affine connection. We use a three-dimensional manifold B to denote a body, whose deformation is described by mapping (2.2), $\phi : B \to \phi(B) \subset \bar{B}$, where $\bar{B}$ is the space in which we imagine B to move. Points in $\bar{B}$ are denoted by $\bar{\mathbf{x}}$ and local coordinates in $\bar{B}$ by $\bar{x}^a$. The tangent map $T\phi : TB \to T\bar{B}$ will be, exclusively in this section, denoted by $\mathbf{F}$ and identified with the damage deformation gradient of ϕ.

According to the concept of damage in continuous media (Kachanov, 1958), one can interpret the mapping ϕ, (2.2), as a damage deformation, if the change of area element $\mathrm{d}\Sigma$ at an arbitrary point $\mathbf{x} \in B$ into the current area element $\mathrm{d}\bar{\Sigma}$ at $\bar{\mathbf{x}} \in \bar{B}$, weakened by a deformation process, cannot be reproduced during the unloading process. A damage state regarded here as a non-holonomicity (due to existing voids, impurities and microcracks) cannot be realized effectively using the body description. Since a damage process is a non-linear anisotropic phenomenon (Fu et al., 1998), it is reasonable to use the fibre bundle structure for its continual modelling, where the geometric structure of damage can be invariantly decomposed into a linear vertical structure and a non-linear horizontal one. In this section, the damage in the medium is described on the tangent bundle over the medium with an affine connection. A fibre bundle approach to an anisotropic damage analysis is presented by Fu et al. (1998).

Assume that an (intially) strained or dislocated state of B can be described by an affine connection ∇ with components Γ^i_{jk} in B. These coefficients can be used to estimate the local dislocated state of the medium caused by the growth of pre-existing microdefects, as well as by the nucleation and growth of new microcracks. By (x^i, y^i) we denote the local induced coordinates in each subset of TB induced from x^i and, by $(\bar{x}^i, \bar{y}^i)$ the induced coordinates in each subset of $T\phi(B) \subset T\bar{B}$. The natural fields of frames on $TB, T\bar{B}$ and coframes on $T^{\star}B, T^{\star}\bar{B}$ are denoted by $(\mathbf{g}_i, \mathbf{z}_i), (\bar{\mathbf{g}}_a, \bar{\mathbf{z}}_a)$ and $(\mathbf{g}^i, \mathbf{z}^i), (\bar{\mathbf{g}}^a, \bar{\mathbf{z}}^a)$, respectively, where $\mathbf{g}_i = \partial/\partial x^i$, $\bar{\mathbf{g}}_a = \partial/\partial \bar{x}^a, \mathbf{g}^i = \mathrm{d}x^i, \bar{\mathbf{g}}^a = \mathrm{d}\bar{x}^a, \mathbf{z}_i = \partial/\partial y^i, \bar{\mathbf{z}}_a = \partial/\partial \bar{y}^a, \mathbf{z}^i = \mathrm{d}y^i$ and $\bar{\mathbf{z}}^a = \mathrm{d}\bar{y}^a$.

Next, we introduce a γ-operation (cf. Yano and Ishihara, 1973) which, for a scalar f, a vector $\mathbf{X}$ and a tensor $\mathbf{A}$ defined on B, is expressed by the relations

$$\gamma f = 0, \qquad \gamma_{\mathbf{x}} f = 0, \tag{4.1}$$

$$\gamma_{\mathbf{x}} \mathbf{A} = x^i A^k_i \mathbf{z}_k, \tag{4.2a}$$

$$\gamma \mathbf{A} = y^i A^k_i \mathbf{z}_k. \tag{4.2b}$$

Applying the γ-operation to the gradient of the function f and put $\nabla_\gamma f = \gamma(\nabla f)$, the horizontal lift f^{H} of f in B to the tangent bundle TB is defined by

$$f^{\mathrm{H}} = y^i \partial_i f - \nabla_\gamma f = \partial f - \nabla_\gamma f = 0, \tag{4.3}$$

where ∂_i denotes the partial derivative with respect to x^i and $\partial f = y^i \partial_i f$. In the same way, the horizontal lift $\mathbf{X}^{\mathrm{H}}$ of the vector $\mathbf{X}$ is given by

$$\mathbf{X}^{\mathrm{H}} = \mathbf{X}^{\mathrm{C}} - \nabla_\gamma \mathbf{X}, \tag{4.4}$$

where $\mathbf{X}^{\mathrm{C}}$, called the complete lift of $\mathbf{X}$, is defined by

$$\mathbf{X}^{\mathrm{C}} = X^i \mathbf{g}_i + \partial_i X^k y^i \mathbf{z}_k. \tag{4.5}$$

In our interpretation, the position vector $\mathbf{X}$ in the medium without distinguished microstructure is prolonged to a position $\mathbf{X}^{\mathrm{C}}$ in the medium with microstructure (cf. Crampin and Pirani, 1986).

In view of Eq. (4.5) and applying Eq. (4.2a) to $\nabla_\gamma \mathbf{X} = \gamma \nabla \mathbf{X} = y^i \nabla_i X^k \mathbf{z}_k$, Eq. (4.4) becomes

$$\mathbf{X}^{\mathrm{H}} = X^i \mathbf{g}_i - \Gamma^i_k X^k \mathbf{z}_i \tag{4.6}$$

with $\Gamma^i_k = y^l \Gamma^i_{lk}$. Here, the original vector $\mathbf{X}$ in B is transported parallely to the induced base in TB.

Corresponding to Eq. (4.6), the horizontal lift ${}^{\mathrm{H}}\mathbf{g}_i$ of $\mathbf{g}_i$ from B into TB takes the form (cf. Eq. (2.8))

$${}^{\mathrm{H}}\mathbf{g}_i = \mathbf{g}_i - \Gamma^k_i \mathbf{z}_k \tag{4.7}$$

and its dual analogue is expressed, using the concept of lifting of a 1-form (Yano and Ishihara, 1973), by

$${}^{\mathrm{H}}\mathbf{g}^i = \mathbf{z}^i + \Gamma^i_k \mathbf{g}^k. \tag{4.8}$$

This is exactly the same result as that given by Eq. (2.9b), which shows the consistency of the horizontal lifting with the idea of connection.

In order to obtain the horizontal lift $\mathbf{F}^{\mathrm{H}}$ of $\mathbf{F}$, we have to introduce the concept of vertical lift of some fields. For $f, \mathbf{X}$ and $\mathbf{A}$ their vertical lifts (Yano and Ishihara, 1973) are defined by the following relations:

$$f^{\mathrm{V}} = f, \quad \mathbf{X}^{\mathrm{V}} = X^k \mathbf{z}_k, \quad \mathbf{A}^{\mathrm{V}} = A^k_i \mathbf{z}_k \otimes \mathbf{g}^i. \tag{4.9a–c}$$

It is seen from Eq. (4.9b) that the vertical lift of the vector $\mathbf{X}$ changes only the basis while its value does not change.

Using (4.9b) for vertical lifts of the bases vectors $\mathbf{g}_i$ and $\mathbf{g}^i$, we have

$${}^{\mathrm{V}}\mathbf{g}_i = \mathbf{z}_i, \qquad {}^{\mathrm{V}}\mathbf{g}^i = \mathbf{g}^i. \tag{4.10}$$

Taking into account Eqs. (4.3) and (4.7)–(4.9a–c), the horizontal lift $\mathbf{F}^{\mathrm{H}}$ of $\mathbf{F}$ is defined as follows:

$$\begin{aligned}\mathbf{F}^{\mathrm{H}} &= (F^a_i \bar{\mathbf{g}}_a)^{\mathrm{H}} \otimes {}^{\mathrm{V}}\mathbf{g}^i + (F^a_i \bar{\mathbf{g}}_a)^{\mathrm{V}} \otimes {}^{\mathrm{H}}\mathbf{g}^i \\ &= F^a_i \bar{\mathbf{g}}_a \otimes \mathbf{g}^i + (\Gamma^j_i F^a_j - \bar{\Gamma}^a_b F^b_i)\bar{\mathbf{z}}_a \otimes \mathbf{g}^i + F^a_i \bar{\mathbf{z}}_a \otimes \mathbf{z}^i,\end{aligned} \tag{4.11a, b}$$

where the quantities $\bar{\Gamma}^b_a = \bar{y}^c \bar{\Gamma}^b_{ca}$ and $\bar{\Gamma}^b_{ca}$ are components of $\overline{\nabla}$ in $\phi(B) \subset \bar{B}$. The representation (4.11b) reveals the influence of the damage or non-holonomicity in the medium on the deformation gradient $\mathbf{F}$.

Under this preparation, we come to define the damage tensor. Let $\mathrm{d}\mathbf{S}$ be the area element vector defined by the cross-product of the two vectors $\mathbf{V}^{\mathrm{H}}$ and $\mathbf{W}^{\mathrm{H}}$ in B

$$\mathrm{d}\mathbf{S} = \mathbf{V}^{\mathrm{H}} \times \mathbf{W}^{\mathrm{H}} = (\mathbf{1} - \mathbf{D}_0)\mathrm{d}\mathbf{\Sigma}, \tag{4.12}$$

where, in view of Eq. (4.6),

$$\mathrm{d}\mathbf{\Sigma} = \mathbf{V} \times \mathbf{W} = V^i W^j \mathbf{g}_i \times \mathbf{g}_j, \tag{4.13}$$

$$\mathbf{D}_0 \mathrm{d}\mathbf{\Sigma} = \mathbf{V} \times \mathbf{W} - \mathbf{V}^{\mathrm{H}} \times \mathbf{W}^{\mathrm{H}} = (\Gamma^k_i V^J W^i - \Gamma^k_i V^i W^j)\mathbf{z}_k \times \mathbf{g}_j - \Gamma^k_i \Gamma^l_j V^i W^j \mathbf{z}_k \times \mathbf{z}_l. \tag{4.14}$$

If a damage state existing in the body B may be regarded as the initial damage, then the tensor $\mathbf{D}_0$ can be identified with the initial damage tensor as a tensorial representation of the non-holonomicity in the damaged medium. In turn, the vector $d\mathbf{\Sigma}_0 = \mathbf{D}_0 d\mathbf{\Sigma}$ represents the damaged or invalidated area element $d\mathbf{\Sigma}$ which, in turn, is identified with a decrease of the load carrying area (cf. Chaboche, 1988b). It is possible to define the damage state of B induced by $\mathbf{F}$ and its influences on the initial damage as well (Fu et al., 1998). In particular, suppose that $d\bar{\mathbf{S}}$ is the element area vector defined by two vectors $\bar{\mathbf{V}}^{\mathrm{H}}$ and $\bar{\mathbf{W}}^{\mathrm{H}}$ in $T\bar{B}$. It is represented, similar to Eq. (4.12), by

$$d\bar{\mathbf{S}} = \bar{\mathbf{V}}^{\mathrm{H}} \times \bar{\mathbf{W}}^{\mathrm{H}} = (\mathbf{1} - \mathbf{D}_{0r})d\bar{\mathbf{\Sigma}}, \tag{4.15}$$

where

$$d\bar{\mathbf{\Sigma}} = \bar{\mathbf{V}} \times \bar{\mathbf{H}} = \bar{V}^a \bar{W}^b \bar{\mathbf{g}}_a \times \bar{\mathbf{g}}_b, \tag{4.16}$$

$$\mathbf{D}_{0r} d\bar{\mathbf{\Sigma}} = \bar{\mathbf{V}} \times \bar{\mathbf{W}} - \bar{\mathbf{V}}^{\mathrm{H}} \times \bar{\mathbf{W}}^{\mathrm{H}} = (\bar{\Gamma}^c_a \bar{V}^b \bar{W}^a - \bar{\Gamma}^c_a \bar{V}^a \bar{W}^b)\,\bar{\mathbf{z}}_c \times \bar{\mathbf{g}}_b - \bar{\Gamma}^c_a \bar{\Gamma}^n_b \bar{V}^a \bar{W}^b\,\bar{\mathbf{z}}_c \times \bar{\mathbf{z}}_n. \tag{4.17}$$

According to $(\mathbf{F}\mathbf{X})^{\mathrm{H}} = \mathbf{F}^{\mathrm{H}}\mathbf{X}^{\mathrm{H}}$, expression (4.15) may be given in the following form:

$$d\bar{\mathbf{S}} = J^{\mathrm{H}}(\mathbf{F}^{\mathrm{H}})^{-\mathrm{T}} d\mathbf{S} = (\mathbf{1} - \mathbf{D}^{\mathrm{H}}_F)d\mathbf{S} = J\mathbf{F}^{-\mathrm{T}}(\mathbf{1} - \mathbf{D}_r)d\mathbf{S}, \tag{4.18}$$

where $J^{\mathrm{H}} = \det \mathbf{F}^{\mathrm{H}}$, $J = \det \mathbf{F}$, and the horizontal damage tensor $(\mathbf{1} - \mathbf{D}^{\mathrm{H}}_F)$ is defined by

$$\mathbf{1} - \mathbf{D}^{\mathrm{H}}_F = J^{\mathrm{H}}(\mathbf{F}^{\mathrm{H}})^{-\mathrm{T}}.$$

The tensor $\mathbf{D}_r$ defined by $(\mathbf{1} - \mathbf{D}_r) = J^{-1}J^{\mathrm{H}}\mathbf{F}^{\mathrm{T}}(\mathbf{F}^{\mathrm{H}})^{-\mathrm{T}}$ in Eq. (4.18) represents the influence of non-holonomicity or defects on the deformation gradient $\mathbf{F}$. In the sequel, it is identified with the additional damage tensor.

From Eqs. (4.12), (4.15) and (4.18) and $d\bar{\mathbf{\Sigma}} = J\mathbf{F}^{-\mathrm{T}}d\mathbf{\Sigma}$, we have

$$\mathbf{1} - \mathbf{D}_{0r} = \mathbf{F}^{-\mathrm{T}}(\mathbf{1} - \mathbf{D}_r)(\mathbf{1} - \mathbf{D}_0)\mathbf{F}^{\mathrm{T}}, \tag{4.19}$$

which shows that the initial damage in B and the additional damage in the process of deformation are transmitted to $\bar{B}$ by $\mathbf{F}$, where the tensor $\mathbf{D}_{0r}$ is called the transferred damage tensor.

Combining Eqs. (4.12) and (4.18), we have

$$d\bar{\mathbf{S}} = (\mathbf{1} - \mathbf{D})d\mathbf{\Sigma}, \tag{4.20}$$

where the total damage tensor, according to Eq. (4.19),

$$\mathbf{1} - \mathbf{D} = J\mathbf{F}^{-\mathrm{T}}(\mathbf{1} - \mathbf{D}_r)(\mathbf{1} - \mathbf{D}_0) = J(\mathbf{1} - \mathbf{D}_{0r})\mathbf{F}^{-\mathrm{T}}. \tag{4.21}$$

Denoting by $\mathbf{D}_{\mathbf{F}}$ the direct damage tensor induced by the deformation $\mathbf{F}$, from Eq. (4.21) one can obtain two equivalent representations of the total damage tensor. The first representation,

$$\mathbf{1} - \mathbf{D} = (\mathbf{1} - \mathbf{D}_{\mathbf{F}})(\mathbf{1} - \mathbf{D}_r)(\mathbf{1} - \mathbf{D}_0), \tag{4.22}$$

defines the total damage to be composed from initial, additional, and direct (deformation-induced) damages. In turn, the second representation,

$$\mathbf{1} - \mathbf{D} = (\mathbf{1} - \mathbf{D}_{0r})(\mathbf{1} - \mathbf{D}_{\mathbf{F}}). \tag{4.23}$$

shows that the total damage can be expressed by direct and transferred damages.

4.2. Constitutive modelling

The problem to find constitutive relations for solids which undergo finite inelastic changes has become a subject of major interest in the last period. To complete the system of field equations presented in Section 3.2, we formulate, in this section, macro- and micro-constitutive equations, which have to be compatible with the principle of material frame indifference, the Clasius–Duhem inequality and the symmetry properties of the material. We restrict our considerations to the constitutive modelling of the Newtonian part of the direct motion.

4.2.1. Micromodel of thermo-inelasticity

Assume that the Helmholtz free energy per unit mass Ψ is the relevant thermodynamic functional to describe the response of the deformation process of the solid. Expanding the state function Ψ with respect to the independent variables, we obtain

$$\begin{aligned}\rho_0\Psi(\mathbf{E}^{\mathrm{h}},\mathbf{E}^{\mathrm{v}},\theta) &= \rho_0\Psi(\mathbf{E}_0^{\mathrm{h}},\mathbf{E}_0^{\mathrm{v}},\theta_0) + \mathbf{S}_0^{\mathrm{h}}\cdot\Delta\mathbf{E}^{\mathrm{h}} + \mathbf{S}_0^{\mathrm{v}}\cdot\Delta\mathbf{E}^{\mathrm{v}} - \rho_0\eta_0\Delta\theta \\ &\quad + \frac{1}{2}\Big\{ {}^{\mathrm{h}}\mathbb{C}[\Delta\mathbf{E}^{\mathrm{h}}]\cdot\Delta\mathbf{E} + {}^{\mathrm{v}}\mathbb{C}[\Delta\mathbf{E}^{\mathrm{v}}]\cdot\Delta\mathbf{E}^{\mathrm{v}} + {}^{\mathrm{h}}{}_{\mathrm{v}}\mathbb{C}[\Delta\mathbf{E}^{\mathrm{h}}]\cdot\Delta\mathbf{E}^{\mathrm{v}} + {}^{\mathrm{v}}{}_{\mathrm{h}}\mathbb{C}[\Delta\mathbf{E}^{\mathrm{v}}]\cdot\Delta\mathbf{E}^{\mathrm{h}} \\ &\quad - \frac{C_V}{\theta_0}\rho_0(\Delta\theta)^2 - C_V\Delta\theta\big[\boldsymbol{\gamma}^{\mathrm{h}}\cdot\Delta\mathbf{E}^{\mathrm{h}} + \boldsymbol{\gamma}^{\mathrm{v}}\cdot\Delta\mathbf{E}^{\mathrm{v}}\big]\Big\} + \cdots, \end{aligned} \tag{4.24}$$

where $\mathbf{S}_0^{\mathrm{h}}$ and $\mathbf{S}_0^{\mathrm{v}}$ are macro- and microstresses, η_0 the entropy evaluated at the equilibrium state $(\mathbf{E}_0^{\mathrm{h}},\mathbf{E}_0^{\mathrm{v}},\theta_0)$, $\Delta\mathbf{E}^{\mathrm{h}} = \mathbf{E}^{\mathrm{h}} - \mathbf{E}_0^{\mathrm{h}},\ldots$ the increments of the dependent variables with respect to the equilibrium state and ${}^{\mathrm{h}}\mathbb{C}$, ${}^{\mathrm{v}}\mathbb{C}$, ${}^{\mathrm{h}}{}_{\mathrm{v}}\mathbb{C}$, ${}^{\mathrm{v}}{}_{\mathrm{h}}\mathbb{C}$ are material tensors defined as partial derivatives of Ψ with respect to the independent variables

$$ {}^{\mathrm{h}}\mathbb{C} = \frac{\partial\mathbf{S}^{\mathrm{h}}}{\partial\mathbf{E}^{\mathrm{h}}}, \quad {}^{\mathrm{v}}\mathbb{C} = \frac{\partial\mathbf{S}^{\mathrm{v}}}{\partial\mathbf{E}^{\mathrm{v}}}, \quad {}^{\mathrm{h}}_{\mathrm{v}}\mathbb{C} = \frac{\partial\mathbf{S}^{\mathrm{h}}}{\partial\mathbf{E}^{\mathrm{v}}}, \quad {}^{\mathrm{v}}_{\mathrm{h}}\mathbb{C} = \frac{\partial\mathbf{S}^{\mathrm{v}}}{\partial\mathbf{E}^{\mathrm{h}}}, $$

where

$$\mathbf{S}^{\mathrm{h}} = \rho_0\frac{\partial\Psi}{\partial\mathbf{E}^{\mathrm{h}}}, \qquad \mathbf{S}^{\mathrm{v}} = \rho_0\frac{\partial\Psi}{\partial\mathbf{E}^{\mathrm{v}}}, \qquad \eta = -\frac{\partial\Psi}{\partial\theta}. \tag{4.25}$$

In relation (4.24), we denote by C_V the specific heat at constant volume,

$$C_V = -\theta\frac{\partial^2\Psi}{\partial\theta^2} = \theta\frac{\partial\eta}{\partial\theta}, \tag{4.26}$$

by γ^{h} and γ^{v} the macro- and micropart of the Grüneisen tensor (cf. Grüneisen, 1912),

$$\boldsymbol{\gamma}^{\mathrm{h}} = \rho_0\left(\theta\frac{\partial^2\Psi}{\partial\theta^2}\right)^{-1}\frac{\partial^2\Psi}{\partial\theta\partial\mathbf{E}^{\mathrm{h}}}, \qquad \boldsymbol{\gamma}^{\mathrm{v}} = \rho_0\left(\theta\frac{\partial^2\Psi}{\partial\theta^2}\right)^{-1}\frac{\partial^2\Psi}{\partial\theta\partial\mathbf{E}^{\mathrm{v}}} \tag{4.27}$$

expressing the interaction between the strain and temperature fields. The tensor γ was first introduced by Grüneisen (1912) as a parameter to describe the proportionality of the volume expansion coefficient α with the specific heat at constant volume C_V.

From Eq. (4.24), the stress tensors and the entropy, Eq. (4.25), can be expressed in the case of harmonic representation as

$$\eta - \eta_0 = \frac{C_V}{\theta_0}\Delta\theta - C_V(\boldsymbol{\gamma}^{\mathrm{h}}\cdot\Delta\mathbf{E}^{\mathrm{h}} + \boldsymbol{\gamma}^{\mathrm{v}}\cdot\Delta\mathbf{E}^{\mathrm{v}}), \tag{4.28}$$

$$\begin{aligned}\mathbf{S}^{\mathrm{h}} - \mathbf{S}_0^{\mathrm{h}} &= {}^{\mathrm{h}}\mathbb{C}[\Delta\mathbf{E}^{\mathrm{h}}] + {}^{\mathrm{h}}_{\mathrm{v}}\mathbb{C}[\Delta\mathbf{E}^{\mathrm{v}}] - C_V\Delta\theta\boldsymbol{\gamma}^{\mathrm{h}}, \\ \mathbf{S}^{\mathrm{v}} - \mathbf{S}_0^{\mathrm{v}} &= {}^{\mathrm{v}}\mathbb{C}[\Delta\mathbf{E}^{\mathrm{v}}] + {}^{\mathrm{v}}_{\mathrm{h}}\mathbb{C}[\Delta\mathbf{E}^{\mathrm{h}}] - C_V\Delta\theta\boldsymbol{\gamma}^{\mathrm{v}}. \end{aligned} \tag{4.29}$$

Relations (4.29) are just the incremental form of the thermo-inelastic constitutive law.

Taking into account that $\mathbf{x}, \mathbf{y}$ and t are independent space–internal space–time variables in the direct motion description, we obtain from Eq. (4.29) or differentiating with respect time, relations (4.25) the rate form of the constitutive equations

$$\begin{aligned}\dot{\mathbf{S}}^{\mathrm{h}} &= {}^{\mathrm{h}}\mathbb{C}[\dot{\mathbf{E}}^{\mathrm{h}}] + {}^{\mathrm{h}}_{\mathrm{v}}\mathbb{C}[\dot{\mathbf{E}}^{\mathrm{v}}] - C_V\dot{\theta}\boldsymbol{\gamma}^{\mathrm{h}},\\ \dot{\mathbf{S}}^{\mathrm{v}} &= {}^{\mathrm{v}}\mathbb{C}[\dot{\mathbf{E}}^{\mathrm{v}}] + {}^{\mathrm{v}}_{\mathrm{h}}\mathbb{C}[\dot{\mathbf{E}}^{\mathrm{h}}] - C_V\dot{\theta}\boldsymbol{\gamma}^{\mathrm{v}},\end{aligned} \tag{4.30}$$

where $C_V\dot{\theta}\boldsymbol{\gamma}^{\mathrm{h}}$ and $C_V\dot{\theta}\boldsymbol{\gamma}^{\mathrm{v}}$ are the changes in the stress state due to the temperature changes. In reality, the field variable $\mathbf{y}$ defining the internal state of the body satisfies the evolution equation (2.17).

4.2.2. Associated macromodel of finite thermo-inelasticity

The micromodel of finite inelasticity presented in Section 4.2.1 can be transformed into a phenomenological model of finite thermo-inelasticity by introducing, besides the free energy Ψ, a second inelastic potential Φ,

$$\Phi = \hat{\Phi}(\mathbf{E}^{\mathrm{h}}, \mathbf{E}^{\mathrm{v}}, \nabla^{\mathrm{h}}\theta, \nabla^{\mathrm{v}}\theta) = \bar{\Phi}(\mathbf{S}^{\mathrm{v}}, \mathbf{E}^{\mathrm{v}}, \nabla^{\mathrm{h}}\theta, \nabla^{\mathrm{v}}\theta), \tag{4.31}$$

subject to the restrictions

$$\Phi \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\Phi = 0, \tag{4.32}$$

where the parameter $\dot{\lambda}$ depending on the deformation history can be called the inelastic consistency parameter.

With the introduction of the inelastic potential (4.31), we are able to determine the microstrain tensor $\mathbf{E}^{\mathrm{v}}$ by an evolution law of the form

$$\dot{\mathbf{E}}^{\mathrm{v}} = \dot{\lambda}\frac{\partial\Phi}{\partial\mathbf{S}^{\mathrm{v}}} \tag{4.33}$$

corresponding to the normality rule resulting from Hill's maximum work principle.

For inelastic loading, when $\dot{\lambda} > 0$, we have $\Phi = 0$ from Eq. (4.33). Then the inelastic consistency condition leads to $\dot{\Phi} = 0$ from which $\dot{\lambda}$ is calculated. After elimination of $\dot{\mathbf{E}}^{\mathrm{v}}$, the set of constitutive equations (4.30) can be transformed into

$$\dot{\mathbf{S}}^{\mathrm{h}} - \dot{\mathbf{S}}^{\mathrm{h}}_{\mathrm{th}} = {}^{\mathrm{IN}}\mathbb{C} \cdot \dot{\mathbf{E}}^{\mathrm{h}}, \tag{4.34}$$

where the inelastic material tensor ${}^{\mathrm{IN}}\mathbb{C}$ is defined by

$${}^{\mathrm{IN}}\mathbb{C} = {}^{\mathrm{h}}\mathbb{C} - D^{-1}\left({}^{\mathrm{h}}_{\mathrm{v}}\mathbb{C} \cdot \frac{\partial\Phi}{\partial\mathbf{S}^{\mathrm{v}}} \otimes \frac{\partial\Phi}{\partial\mathbf{S}^{\mathrm{v}}} \cdot {}^{\mathrm{v}}_{\mathrm{h}}\mathbb{C}\right) \tag{4.35}$$

with the denominator

$$D = \frac{\partial\Phi}{\partial\mathbf{S}^{\mathrm{v}}} \cdot {}^{\mathrm{v}}\mathbb{C} \cdot \frac{\partial\Phi}{\partial\mathbf{S}^{\mathrm{v}}} + \frac{\partial\Phi}{\partial\mathbf{E}^{\mathrm{v}}} \cdot \frac{\partial\Phi}{\partial\mathbf{S}^{\mathrm{v}}}.$$

The thermal stress rate $\dot{\mathbf{S}}^{\mathrm{h}}_{\mathrm{th}}$ in Eq. (4.34) is defined by

$$\dot{\mathbf{S}}^{\mathrm{h}}_{\mathrm{th}} = -C_V\dot{\theta}\boldsymbol{\gamma}^{\mathrm{h}} + D^{-1}{}^{\mathrm{v}}_{\mathrm{h}}\mathbb{C} \cdot \frac{\partial\Phi}{\partial\mathbf{S}^{\mathrm{v}}}\left(\frac{\partial\Phi}{\partial\mathbf{S}^{\mathrm{v}}}C_V\dot{\theta}\boldsymbol{\gamma}^{\mathrm{v}} - \frac{\partial\Phi}{\partial\boldsymbol{\Theta}}\dot{\boldsymbol{\Theta}}\right), \tag{4.36}$$

where $\boldsymbol{\Theta} = (\nabla^{\mathrm{h}}\theta, \nabla^{\mathrm{v}}\theta)$.

If $\mathbf{A} = {}^{\mathrm{h}}_{\mathrm{v}}\mathbb{C} \cdot \partial\Phi/\partial\mathbf{S}^{\mathrm{v}}$ can be approximated by $\mathbf{A} = \mathrm{tr}\,\mathbf{A}\mathbf{1}$ or ${}^{\mathrm{h}}\mathbb{C}$ has the classical form then Eq. (4.35) can be rewritten as

$$ {}^{\mathrm{IN}}\mathbb{C} = \left[1 - D^{-1}\left({}^{\mathrm{h}}_{\mathrm{v}}\mathbb{C} \cdot \frac{\partial \Phi}{\partial \mathbf{S}^{\mathrm{v}}}\right) \cdot {}^{\mathrm{h}}\mathbb{C}^{-1} \cdot \left(\frac{\partial \Phi}{\partial \mathbf{S}^{\mathrm{v}}} \cdot {}^{\mathrm{v}}_{\mathrm{h}}\mathbb{C}\right)\right]{}^{\mathrm{h}}\mathbb{C} = (1-k)^{\mathrm{h}}\mathbb{C}, \tag{4.37} $$

where

$$ k = D^{-1}\left({}^{\mathrm{h}}_{\mathrm{v}}\mathbb{C} \cdot \frac{\partial \Phi}{\partial \mathbf{S}^{\mathrm{v}}}\right) \cdot {}^{\mathrm{h}}\mathbb{C}^{-1} \cdot \left(\frac{\partial \Phi}{\partial \mathbf{S}^{\mathrm{v}}} \cdot {}^{\mathrm{v}}_{\mathrm{h}}\mathbb{C}\right) $$

is a scalar-valued variable.

Restricting the Helmholtz free energy functional to the form

$$ \Psi = \bar{\Psi}(\mathbf{E}^{\mathrm{h}}(d), \theta) \equiv \hat{\Psi}(\mathbf{E}^{\mathrm{h}}, d, \theta), \tag{4.38} $$

where $d = d(\mathbf{X}, t)$ is the crack density we define stress measures of the second Piola–Kirchhoff type

$$ \mathbf{S}^{\mathrm{h}} = \rho_0 \frac{\partial \Psi}{\partial \mathbf{E}^{\mathrm{h}}}, \qquad S_d = \rho_0 \frac{\partial \Psi}{\partial d} \tag{4.39} $$

with S_d considered as scalar-valued stress variable, power conjugate to the kinematical crack density function d. If we assume that the dissipation potential of a damage evolution

$$ \Phi = \hat{\Phi}(\mathbf{E}^{\mathrm{h}}, d, \nabla\theta), \tag{4.40} $$

then the inelastic material tensor ${}^{\mathrm{IN}}\mathbb{C}$ reduces to form (4.37). This case can be treated as a generalization of Kachanov's tangent operator with a scalar-valued damage variable

$$ k = \frac{\frac{\partial \Phi}{\partial S_d} \frac{\partial \mathbf{E}^{\mathrm{h}}}{\partial d} \cdot {}^{\mathrm{h}}\mathbb{C} \cdot \frac{\partial \mathbf{E}^{\mathrm{h}}}{\partial d}}{C_d \frac{\partial \Phi}{\partial S_d} + \frac{\partial \Phi}{\partial d}}, $$

which varies from 0 in the undamaged material to 1 at fracture.

4.3. The continuum $\mathscr{B}$ with distinguished cracks

In this section, we find the extremum of a modified functional I_{Σ_t}, which includes the evolving crack surface Σ_t inside the body. We do not pose a priori any restriction on the topography of cracks or of the crack process of the body. We rather consider in a bounded, connected, open region $G \subseteq \mathscr{B}$ the family of possible cracked states. To each member of this family is assigned the surface energy created by the evolution process of cracks. We have to consider the crack propagation problem by finding extrema of the functional

$$ I_{\Sigma_t} = \int_{G_t}\int_T \mathscr{L}_t(\mathbf{x}, t, \mathbf{X}, \mathbf{F}^{\mathrm{h}}, \dot{\mathbf{X}})\,\mathrm{d}V\,\mathrm{d}t + \int_{\partial G_t}\int_T (\mathbf{t}^{\mathrm{h}} \cdot \mathbf{u}^{\mathrm{h}} + \mathbb{t}^{\mathrm{h}} \cdot \mathbb{u}^{\mathrm{h}})\,\mathrm{d}S\,\mathrm{d}t + \int_{\Sigma_t}\int_T \mathscr{L}_\Sigma(\mathbf{x}, t, \mathbf{X}, \dot{\mathbf{X}})\,\mathrm{d}S\,\mathrm{d}t, \tag{4.41} $$

where $G_t = G \setminus \Sigma_t$, $\mathbf{u}^{\mathrm{h}}$ and $\mathbb{u}^{\mathrm{h}}$ are virtual displacements for a given load increment. This functional represents the total energy of the body for a given crack surface Σ_t and a given loading process. The function $\mathscr{L}_\Sigma$ can be identified with the energy of macro- and microcracks, which arises from a non-equilibrium state of material particles in the neighbourhood of surface points. This energy is assumed to be composed of the cohesive energy of bonds and the energy of dissipation due to macro- and microcrack growth at the initial stage of the deformation process. For further load increments, $\mathscr{L}_\Sigma$ accounts for the work of configurational and deformational forces as well as the kinetic energy of points in the vicinity of a crack tip. One has to emphasize that the energy density function $\mathscr{L}_\Sigma$ can, in general, be discontinuous along certain curves lying on Σ_t.

The functional I_{Σ_t} of Eq. (4.41), differs from the functional I_t of Eq. (3.1) only by the term $\int_{\Sigma_t}\int_T \mathscr{L}_{\Sigma_t} \mathrm{d}S\,\mathrm{d}t$, which has no influence on the extremal properties of the functional. Since an extremum of I_{Σ_t} can occur only along the solutions of the Euler–Lagrange equations (3.18) and (3.21), the variation δI_{Σ_t} takes the form

$$\delta I_{\Sigma_t} = \int_{\partial G_t}\int_T \left[(\mathbf{t}^{\mathrm{h}} - \mathbf{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}}) \cdot \delta\mathbf{X} + (\mathbb{t}^{\mathrm{h}} - \mathbb{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}}) \cdot \delta\mathbf{x}\right] \mathrm{d}S\,\mathrm{d}t + \int_{\Sigma_t}\int_T (\nabla^{\mathrm{h}}\mathscr{L}_\Sigma \cdot \delta\mathbf{x} + \overline{\nabla}^{\mathrm{h}}\mathscr{L}_\Sigma \cdot \delta\mathbf{X}) \mathrm{d}S\,\mathrm{d}t, \tag{4.42}$$

where

$$\overline{\nabla}^{\mathrm{h}}_a \mathscr{L}_\Sigma = \frac{\partial \mathscr{L}_\Sigma}{\partial X^a} - N^b_a \frac{\partial \mathscr{L}_\Sigma}{\partial \dot{X}^b} \tag{4.43}$$

is the surface force–momentum vector [8] and $\nabla^{\mathrm{h}}\mathscr{L}_\Sigma$ the surface inhomogeneity force, all distributed over Σ_t. The Eulerian covariant operator $\overline{\nabla}^{\mathrm{h}}_i$ used in Eq. (4.43) was introduced, according to the Finslerian geometry methodology, in the form

$$\overline{\nabla}^{\mathrm{h}}_a(\cdot) = \frac{\partial(\cdot)}{\partial X^a} - N^b_a \frac{\partial(\cdot)}{\partial \dot{X}^b}. \tag{4.44}$$

Calculating this variation we assumed that the action integral has been restricted to the integral curves of the Euler–Lagrange equations, because an extremum can be attained only along these solutions.

The boundary conditions, which follow directly from Eq. (4.42), are defined as follows. If the variations $\delta\mathbf{X}$ and $\delta\mathbf{x}$, respectively, coincide on Σ_t in both integrals of Eq. (4.42), then the necessary conditions for Eq. (4.42) take the form

$$\left[\mathbb{t}^{\mathrm{h}} - \mathbb{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}}\right]_{\Sigma_t} \cdot \delta\mathbf{x}_\Sigma + \left[\mathbf{t}^{\mathrm{h}} - \mathbf{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}}\right]_{\Sigma_t} \cdot \delta\mathbf{X}_\Sigma + \nabla^{\mathrm{h}}\mathscr{L}_\Sigma \cdot \delta\mathbf{x}_\Sigma + \overline{\nabla}^{\mathrm{h}}\mathscr{L}_\Sigma \cdot \delta\mathbf{X}_\Sigma = 0. \tag{4.45}$$

If the boundary points of Σ_t can move along the curve $\mathbf{X} = \mathbb{X}_t(\mathbf{x})$, where $\mathbf{X}$ is the function of a parameter s, then introducing $\delta\mathbf{X} = \nabla^{\mathrm{h}}\mathbb{X}_t\delta\mathbf{x}$ into Eq. (4.45), we obtain

$$[\mathscr{L}_t\mathbf{1}^{\mathrm{h}} + ((\mathbf{F}^{\mathrm{h}})^{\mathrm{T}} - (\nabla^{\mathrm{h}}\mathbb{X}_t)^{\mathrm{T}})\mathbf{T}^{\mathrm{h}}]\mathbf{n}^{\mathrm{h}} + \nabla^{\mathrm{h}}\mathbb{X}_t\overline{\nabla}^{\mathrm{h}}\mathscr{L}_\Sigma + \nabla^{\mathrm{h}}\mathscr{L}_\Sigma = 0 \tag{4.46}$$

after neglecting the deformational tractions $\mathbf{t}$ and the configurational tractions $\mathbb{t}$ for internal boundaries of $\mathscr{B}$ during its deformation processes. In practice, it is rather difficult to specify ad hoc such conditions. Condition (4.46) represents the local equilibrium condition with a dependency of the possible crack gradient $\nabla^{\mathrm{h}}\mathbb{X}_t$ on the deformation gradient $\mathbf{F}^{\mathrm{h}}$. It enables one to distinguish the actual crack path $\mathbb{X}_t$ from the thermodynamically admissible paths of the moving crack. In the next step, from Eq. (4.46) one can obtain $\mathbb{X}_t$ and $\nabla^{\mathrm{h}}\mathbb{X}_t$, if we impose on Σ_t suitable boundary and initial conditions for $\overline{\nabla}^{\mathrm{h}}\mathscr{L}_\Sigma$ and $\nabla^{\mathrm{h}}\mathscr{L}_t$. From the calculated $\mathbb{X}_t, \dot{\mathbb{X}}_t, \ldots$ at any load increment, one can update the surface energy

$$\mathscr{L}_{\Sigma_t}(\mathbf{x}_\Sigma, \mathbb{X}_t, \dot{\mathbb{X}}_t)$$

and, according to Eq. (4.43), define the surface–momentum $\overline{\nabla}^{\mathrm{h}}\mathscr{L}_\Sigma$ on Σ_t. One should stress the fact that we are dealing with an evolution of crack surfaces Σ_t, i.e.,

$$\Sigma_{t_2} = \mathbb{X}_t(\cdot, \Sigma_{t_1}) \quad \text{for } t_2 > t_1.$$

[8] Note that the case $\mathbf{N}_\Gamma \equiv \mathbf{0}$ leads here to the decoupling of force and momentum vector and, in effect, to their classical definitions, cf. Eqs. (3.8a,b) and (3.12a,b).

In general, the cited form of the surface energy is not given *a priori*, but it is assumed to be originated from the formation of new crack surfaces. Of course, its strong dependence on the orientation cannot be omitted here.

4.3.1. Crack growth criterion

A (micro)crack placed in the stress intensity field evolves because of an existing driving force arising from a local decrease of the free energy in the body. This force depends on the configuration of the (micro)cracks and other defect arrangements. From the thermodynamics of crack propagation it follows that the crack begins to evolve (to grow), if the driving force, defined by the difference between the increase of the free energy and the work done by the applied stress, is not smaller than the fracture threshold value. This force depends on the configuration of the (micro)cracks and other defect arrangements. The structure of the deformed material has therefore a decisive influence not only on the crack nucleation, but also on the resulting macrocrack formation, strictly correlated with the strength of the material. Such information about the evolution of the internal state of the body can be included in the connection coefficients.

Consideration of crack dynamics requires the analysis of irreversible processes in a zone surrounding the crack tip. For the moving defect, which models the tip of a crack, the total energy release rate is used to define the driving force. In this situation, the crack dynamics requires an investigation of the irreversible processes in a zone surrounding the crack tip. Following this line of thinking (Stumpf and Le, 1990, 1992; Gurtin and Podio-Guidugli, 1998), we introduce the configurational dynamical tip traction which, after identifying $-\mathscr{L}_t$ with the total mechanical energy, i.e., the inelastic energy Ψ plus the kinetic energy K measured relative to the deformed (micro)crack tip, takes the well-known form

$$\mathbf{j} = \oint_{\text{tip}} [(\Psi + K)\mathbf{1}^{\text{h}} - (\mathbf{F}^{\text{h}})^{\text{T}}\mathbf{T}^{\text{h}}]\mathbf{n}^{\text{h}}. \tag{4.47}$$

The tip intergal, $\oint_{\text{tip}}(\cdot)\mathbf{n}^{\text{h}}$, used above, represents an integral around an infinitesimal sphere-type surface (an infinitesimal loop in the case of planar cracks) surrounding the crack tip, with $\mathbf{n}^{\text{h}}$ representing the outward unit normal to the sphere (the loop) (cf. Gurtin and Shvartsman, 1997). The energy release rate is then defined as the scalar product

$$J \equiv \boldsymbol{v} \cdot \mathbf{j} = \boldsymbol{v} \cdot \oint_{\text{tip}} [(\Psi + K)\mathbf{1}^{\text{h}} - (\mathbf{F}^{\text{h}})^{\text{T}}\mathbf{T}^{\text{h}}]\mathbf{n}^{\text{h}}, \tag{4.48}$$

where $\boldsymbol{v}$ is the direction of propagation of the (micro)crack tip. One should stress the fact that in our considerations, the concept of a crack tip is more general than the one usually used in the literature, because any point of the boundary $\partial\Sigma_t$ can be treated as the crack tip. The direction of motion of $\partial\Sigma_t$, which depends on the actual constraints of the surrounding medium, can be described in terms of the crack direction of a single (micro)crack or the mean value of the microcrack directions in the representative volume.

The entropy production is used to identify the driving force power conjugate to the crack evolution rate in the physical process. The scalar value of the driving force f for a crack propagation can be defined as the difference between the strain energy release rate J and the surface energy rate ψ of the crack surface at the tip,

$$f = J - \psi \geqslant 0, \qquad \psi = \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\Sigma_t} \mathscr{L}_{\Sigma_t}(\mathbf{x}_\Sigma, \mathbb{X}_t, \dot{\mathbb{X}}_t)\mathrm{d}S \tag{4.49}$$

as a necessary condition for the crack growth. The crack propagation direction $\boldsymbol{v} = \Delta\mathbf{n}^{\text{h}}_{\Sigma}$ must satisfy the dissipation inequality at every increment of loads. Moreover, in the definition of ψ, a parameter τ can represent a characteristic length of the crack or may denote the time, if it is physically justified.

To define the crack growth criterion for the kinking of cracks, we have to redefine the strain energy release rate J and the surface energy rate ψ in Eq. (4.49). First, consider the first variation of Eq. (4.41), which in view of the condition $\delta I_{\Sigma_t} \geqslant 0$ valid for all families of admissible fields $(\mathbf{X}, \mathbf{F}^{\mathrm{h}})$ (cf. Stumpf and Le, 1990), leads to the following inequality

$$\mathbb{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}}\big|_{\Sigma_t} \cdot \delta\mathbf{x}_\Sigma + \mathbf{T}^{\mathrm{h}}\mathbf{n}^{\mathrm{h}}\big|_{\Sigma_t} \cdot \delta\mathbf{X}_\Sigma \geqslant \mathbb{t}^{\mathrm{h}}\big|_{\Sigma_t} \cdot \delta\mathbf{x}_\Sigma + \mathbf{t}^{\mathrm{h}}\big|_{\Sigma_t} \cdot \delta\mathbf{X}_\Sigma + \nabla^{\mathrm{h}}\mathscr{L}_\Sigma \cdot \delta\mathbf{x}_\Sigma + \overline{\nabla}^{\mathrm{h}}\mathscr{L}_\Sigma \cdot \delta\mathbf{X}_\Sigma. \tag{4.50}$$

If the corner points of Σ_t can move along the curve $\hat{\mathbf{X}}_\Sigma = \hat{\mathbb{X}}_t(\hat{\mathbf{x}}_\Sigma)$ with $\nabla^{\mathrm{h}}\hat{\mathbb{X}}_t = \hat{\mathbf{F}}^{\mathrm{h}}$, where $\hat{\mathbf{X}}_2$ is the function of a parameter $\hat{s}$, then on the basis of Eq. (4.50), one can define

$$J = \oint_{\mathrm{tip}} \left\{ (\Psi + K)\mathbf{1}^{\mathrm{h}} + [(\hat{\mathbf{F}}^{\mathrm{h}})^{\mathrm{T}} - (\mathbf{F}^{\mathrm{h}})^{\mathrm{T}}]\mathbf{T}^{\mathrm{h}} \right\}_{\Sigma_t} \mathbf{n}^{\mathrm{h}} \cdot \boldsymbol{v},$$

$$\psi = \int_{\Sigma_t} \{ [\mathbb{t}^{\mathrm{h}} + (\hat{\mathbf{F}}^{\mathrm{h}})^{\mathrm{T}}\mathbf{t}^{\mathrm{h}}]_{\Sigma_t} + \nabla^{\mathrm{h}}\mathscr{L}_\Sigma + (\hat{\mathbf{F}}^{\mathrm{h}})^{\mathrm{T}}\overline{\nabla}^{\mathrm{h}}\mathscr{L}_\Sigma \} \cdot \boldsymbol{v}\,\mathrm{d}S.$$

From comparison, it follows that the definitions of J and ψ include additional terms resulting from the corner point motions $\hat{\mathbf{X}}_\Sigma$ and the assumed tractions on Σ_t.

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